Practice Final, Math 235 Spring 2009

1. Consider the matrix
$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \\ -2 & 1 & -3 \end{pmatrix}.$$
Let $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$. Find equations in b_1, b_2, b_3, b_4 so that the equation $Ax = b$ can be solved. Find a basis of the image of A.

2. True or false? Justify.
   - If W is a subspace of an n-dimensional vector space V and $dim(W) = n$, the $W = V$.
   - There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose kernel has dimension 1.
   - Let $\mathbb{R}^{2\times 2}$ be the vector space of 2×2-matrices. The function $det : \mathbb{R}^{2\times 2} \to \mathbb{R}$, which maps a matrix A to its determinant $det(A)$, is linear.
   - Let P be the vector space of polynomials in x, and let
   $$W = \left\{ p(x) : xp(x) - 2\int_0^1 p(t)t = 0 \right\}$$
   . Then W is a subspace of P.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\left(\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right) = \left[\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}\right]$ and $T\left(\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\right) = \left[\begin{pmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}\right]$.
   - (a) Calculate $T\left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\right)$.
   - (b) Find the matrix A which corresponds to the transformation $T \circ T$.
   - (c) For which k is T^k the identity transformation? Justify.

4. Let V be the vector space spanned by the functions e^x, xe^x, and x^2e^x, so that V has a basis $\mathfrak{B} = (e^x, xe^x, x^2e^x)$.
   - What is the dimension of V?
   - Let $T : V \to V$ be the linear transformation $T(f(x)) = f'(x)$. Find the matrix of T with respect to the basis $\mathfrak{B}$. Is T invertible?

5. Let V be the vector space of all upper triangular 2×2 matrices. Define the linear transformation $T : V \to V$ as

$$T \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = aI_2 + bP + cP^2,$$

where $P = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

- Find the matrix of T with respect to the basis

$$\mathfrak{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

- Find bases of the image and kernel of T.

6. Compute the determinant of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{pmatrix}$$

7. Let $A = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$.

- Calculate the eigenvalues and eigenvectors of A.
- Check that the eigenvectors $\vec{v}_1$, $\vec{v}_2$ that you calculated above form a basis of $\mathbb{R}^2$.
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map whose matrix with respect to the standard basis is A. Find its matrix with respect to the new basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$.

8. Solve the following differential equation:

$$2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0$$

What is the dimension of its space of solutions?

9. Two interacting populations of coyotes and roadrunners can be modeled by the recursive equations:

$$\begin{pmatrix} c(t+1) \\ r(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 0.75 \\ -1.5 & 2.25 \end{pmatrix} \begin{pmatrix} c(t) \\ r(t) \end{pmatrix}.$$

Find $c(5)$ and $r(5)$ given the initial populations $c(0) = 500$, $r(0) = 700$. What are the limiting values of $c(t)$, $r(t)$?