1. Consider the matrix 
\[ A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \\ -2 & 1 & -3 \end{pmatrix}. \]

Let \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \). Find equations in \( b_1, b_2, b_3, b_4 \) so that the equation \( Ax = b \) can be solved. Find a basis of the image of \( A \).

2. True or false? Justify.

- If \( W \) is a subspace of an \( n \)-dimensional vector space \( V \) and \( \dim(W) = n \), the \( W = V \).
- There exists a linear transformation \( T : \mathbb{R}^5 \to \mathbb{R}^3 \) whose kernel has dimension 1.
- Let \( \mathbb{R}^{2 \times 2} \) be the vector space of \( 2 \times 2 \)-matrices. The function \( \det : \mathbb{R}^{2 \times 2} \to \mathbb{R} \), which maps a matrix \( A \) to its determinant \( \det(A) \), is linear.
- Let \( P \) be the vector space of polynomials in \( x \), and let 
\[ W = \left\{ p(x) : xp(x) - 2 \int_0^1 p(t)t \, dt = 0 \right\}. \]

Then \( W \) is a subspace of \( P \).

3. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) with \( T \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} \sqrt{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \) and \( T \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -\sqrt{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \).

- (a) Calculate \( T \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \).
- (b) Find the matrix \( A \) which corresponds to the transformation \( T \circ T \).
- (c) For which \( k \) is \( T^k \) the identity transformation? Justify.

4. Let \( V \) be the vector space spanned by the functions \( e^x, xe^x, \) and \( x^2 e^x \), so that \( V \) has a basis \( \mathcal{B} = (e^x, xe^x, x^2 e^x) \).

- What is the dimension of \( V \)?
- Let \( T : V \to V \) be the linear transformation \( T(f(x)) = f'(x) \). Find the matrix of \( T \) with respect to the basis \( \mathcal{B} \). Is \( T \) invertible?
5. Let \( V \) be the vector space of all upper triangular \( 2 \times 2 \) matrices. Define the linear transformation \( T : V \rightarrow V \) as
\[
T \left( \begin{array}{cc}
a & b \\
0 & c \\
\end{array} \right) = aI_2 + bP + cP^2,
\]
where \( P = \left( \begin{array}{cc}
1 & 2 \\
0 & 3 \\
\end{array} \right) \).

- Find the matrix of \( T \) with respect to the basis \( \mathfrak{B} = \left\{ \left( \begin{array}{cc}
1 & 0 \\
0 & 0 \\
\end{array} \right), \left( \begin{array}{cc}
0 & 1 \\
0 & 0 \\
\end{array} \right), \left( \begin{array}{cc}
0 & 0 \\
0 & 1 \\
\end{array} \right) \right\} \).
- Find bases of the image and kernel of \( T \).

6. Compute the determinant of the following matrix:
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 3 & 3 \\
2 & 2 & 5
\end{pmatrix}
\]

7. Let \( A = \left( \begin{array}{cc}
0.4 & 0.6 \\
0.6 & 0.4 \\
\end{array} \right) \).

- Calculate the eigenvalues and eigenvectors of \( A \).
- Check that the eigenvectors \( \vec{v}_1, \vec{v}_2 \) that you calculated above form a basis of \( \mathbb{R}^2 \).
- Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear map whose matrix with respect to the standard basis is \( A \). Find its matrix with respect to the new basis \( \mathfrak{B} = \{ \vec{v}_1, \vec{v}_2 \} \).

8. Solve the following differential equation:
\[
2 \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 2y = 0
\]
What is the dimension of its space of solutions?

9. Two interacting populations of coyotes and roadrunners can be modeled by the recursive equations:
\[
\begin{pmatrix}
c(t+1) \\
r(t+1)
\end{pmatrix} = \begin{pmatrix}
0 & 0.75 \\
-1.5 & 2.25
\end{pmatrix} \begin{pmatrix}
c(t) \\
r(t)
\end{pmatrix}.
\]
Find \( c(5) \) and \( r(5) \) given the initial populations \( c(0) = 500, r(0) = 700 \). What are the limiting values of \( c(t), r(t) \)?