235 Common midterm review Questions

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(1) Let m and n be positive integers and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$. What does it mean to say that a vector $\mathbf{v}$ is a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_m$?

(2) Is the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$ a linear combination of the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$?

(3) Let m and n be positive integers and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$. What is meant by the *span* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$?

(4) In each case, give a precise geometric description of the span of the given vectors.

(a) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

(b) $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

(e) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$.

(5) Let m and n be positive integers and $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. What does it mean to say that T is a *linear transformation*? If T is a linear transformation, what is meant by the (standard) matrix A of T? How many rows and columns does A have? How are the columns of A determined by the linear transformation T?

(6) In each case, determine whether the function T is a linear transformation, and if so compute its (standard) matrix.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+1 \\ x \end{pmatrix}$.

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ x+y+z \end{pmatrix}$.

(c) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation about the origin through angle $\pi/4$ counterclockwise.

(d) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by reflection in the yz-plane.

(e) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by orthogonal projection onto the line $y = x$.

(7) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. What is the *kernel* of T? What is the *image* of T? Given the matrix A of T, how can we express the kernel of T as the span of a set of vectors? What about the image?

(8) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\mathbf{v} \in \mathbb{R}^n$. What is the relation between the set of vectors

$$\{\mathbf{w} \in \mathbb{R}^n \mid T(\mathbf{w}) = T(\mathbf{v})\} \subset \mathbb{R}^n$$

and the kernel of T?

(9) In each case, describe the kernel and the image of the linear transformation T as the span of a set of vectors.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by orthogonal projection onto the plane $\Pi \subset \mathbb{R}^3$ with equation $x + 2y + 3z = 0$.

(c) $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ the linear transformation with standard matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{pmatrix}.$$

(10) Let m and n be positive integers and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ a set of vectors in $\mathbb{R}^n$. What does it mean to say that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are *linearly independent*? If they are linearly independent, what can you say about m and n?

(11) In each case, determine whether the given set of vectors is linearly independent.

(a) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$

(b) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 12 \\ 37 \end{pmatrix}.$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 13 \\ 7 \end{pmatrix}.$

(d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}.$

(12) Let m and n be positive integers. What does it mean to say that a subset $W \subset \mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$? (There is a list of 3 conditions which must be satisfied.)

(13) In each case, determine whether the given subset W is a subspace.

(a) $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x + y = 1 \right\} \subset \mathbb{R}^2$

(b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + 2y + 3z = 0 \right\} \subset \mathbb{R}^3$.

(c) $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; y \geq 2x \right\} \subset \mathbb{R}^2$.

(d) $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and (i) $W = \ker(T) \subset \mathbb{R}^n$, (ii) $W = \text{image}(T) \subset \mathbb{R}^m$.

(e) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are vectors in $\mathbb{R}^n$, and $W = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \subset \mathbb{R}^n$ is the span of the set of vectors.

(f) $W \subset \mathbb{R}^n$ is the set of solutions of a system of homogeneous linear equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0 \text{ for } i = 1, \ldots, m$$

(Here the coefficient a_{ij} of x_j in the ith equation is some given real number for each i and j, and each equation has no constant term (the equations are *homogeneous*).)

(14) Let m and n be positive integers. Let $W \subset \mathbb{R}^n$ be a subspace. What does it mean to say that a list of vectors $\mathbf{v}_1, \ldots \mathbf{v}_m$ in $\mathbb{R}^n$ is a *basis* of W? What is meant by the *dimension* of W? Given a linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$, how can we compute a basis of the kernel and the image of T?

(15) In each case, determine a basis of the subspace W. Use your answer to determine the dimension of W.

(a) $W \subset \mathbb{R}^3$ is the plane with equation $x + 2y + 4z = 0$.

(b) W is the kernel of the linear transformation $T \colon \mathbb{R}^5 \to \mathbb{R}^3$ with standard matrix

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

(c) W is the image of the linear transformation $T \colon \mathbb{R}^4 \to \mathbb{R}^3$ with standard matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 1 & 4 \end{pmatrix}.$$

(d) $W = \mathbb{R}^n$.

(16) Let m and n be positive integers and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ a set of vectors in $\mathbb{R}^n$. What can you say about m and n if the vectors (a) are linearly independent? (b) span $\mathbb{R}^n$? (c) are a basis of $\mathbb{R}^n$?

(17) Let m and n be positive integers. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. What is meant by the rank of T? (Give two answers: one suitable for computational purposes in terms of the echelon form of the matrix of T, and another more conceptual in terms of the dimension of a subspace determined by T.) What is the rank nullity formula for the linear transformation T?

(18) What can you say about the dimension of the kernel and image of T in the following cases?

(a) $T\colon \mathbb{R}^8 \to \mathbb{R}^3$.

(b) $T\colon \mathbb{R}^4 \to \mathbb{R}^7$.

(c) $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

(19) Let m and n be positive integers. Let $W \subset \mathbb{R}^n$ be a subspace of dimension m. What can you say about m and n? Let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a basis of W (the number of elements in the basis equals m, the dimension of W, by the definition of dimension). For $\mathbf{v} \in W$ a vector, what is the definition of the $\mathcal{B}$-coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$?

(20) Let $W \subset \mathbb{R}^3$ be the plane given by the equation $x + y + z = 0$. Let $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(a) Explain why $\mathcal{B}$ is a basis of W.

(b) Compute the vector $\mathbf{w} \in W$ with $\mathcal{B}$-coordinate vector $[\mathbf{w}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

(c) Let $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix}$. Show that the vector $\mathbf{v}$ lies in W and compute its $\mathcal{B}$-coordinate vector $[\mathbf{v}]_{\mathcal{B}}$.

(21) Let $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(a) Explain why $\mathcal{B}$ is a basis of $\mathbb{R}^2$.

(b) Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the invertible linear transformation defined by $L(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$. What is the standard matrix of the inverse L^{-1}? (Note: No computation is required!) Use your answer to compute the standard matrix of L.

(22) Let n be a positive integer. Let $\mathcal{B}$ be a basis of $\mathbb{R}^n$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation. What is the definition of the $\mathcal{B}$-matrix of T? How are the columns of the $\mathcal{B}$-matrix determined by the linear transformation T? If A is the standard matrix of T and B is the $\mathcal{B}$-matrix of T, how are A and B related?

(23) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Determine the $\mathcal{B}$-matrix B of T.

(24) For each of the following linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^n$, determine a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix of T is diagonal, and compute this matrix.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $y = 2x$.

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by orthogonal projection onto the plane with equation $2x + y + z = 0$.

(c) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by rotation about the axis with equations $x = y = z$ through angle π radians.

(25) What is a linear space (or vector space)? Make sure you understand the generalizations of the basic notions of linear algebra from $\mathbb{R}^n$ to linear spaces, including span, linear independence, basis, dimension, subspace, linear transformation, kernel, image, etc.

(26) For each of the following linear spaces write down a basis. Use your answer to compute the dimension.

(a) The linear space $\mathcal{P}_4$ of polynomials of degree ≤ 4.

(b) The linear space $\mathbb{R}^{2\times 3}$ of 2×3 matrices.

(c) $\mathcal{P}_n$.

(d) $\mathbb{R}^{m\times n}$.

(27) Which of the following subsets $W \subset V$ are subspaces of the given linear space V? If W is a subspace, find a basis.

(a) $V = \mathcal{P}_2$, $W = \{f(x) \mid f(1) = 0\} \subset \mathcal{P}_2$.

(b) $V = \mathbb{R}^{2\times 2}$, $W = \{X \mid AX = XA\} \subset \mathbb{R}^{2\times 2}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(c) $V = \mathcal{P}_4$, $W = \{f(x) \mid f(-x) = -f(x)\} \subset \mathcal{P}_4$.

(28) In each case, determine whether the given function $T\colon V \to W$ is a linear transformation from the linear space V to the linear space W. If T is linear determine a basis of the kernel and the image of T.

(a) $T\colon \mathcal{P}_3 \to \mathcal{P}_3$, $T(f(x)) = f'(x)$, the derivative of $f(x)$.

(b) $T\colon \mathcal{P}_2 \to \mathbb{R}^2$, $T(f(x)) = \begin{pmatrix} f(1) \\ f(2) \end{pmatrix}$.

(c) $T\colon \mathcal{P}_2 \to \mathcal{P}_2$, $T(f(x)) = xf'(x) - f(x)$.

(29) Let V and W be (finite dimensional) linear spaces and $T\colon V \to W$ a linear transformation. We say T is *invertible* or an *isomorphism* if T has an inverse $T^{-1}\colon W \to V$. If T is an isomorphism then $\dim V = \dim W$ ("an invertible matrix is square"). Conversely, if $\dim V = \dim W$, how can we determine whether T is an isomorphism in terms of (a) the kernel of T and (b) the image of T?

(30) Which of the following linear transformations are isomorphisms?

(a) $T\colon \mathcal{P}_3 \to \mathbb{R}^2$, $T(f(x)) = \begin{pmatrix} f(3) \\ f(5) \end{pmatrix}$.

(b) $T\colon \mathcal{P}_2 \to \mathcal{P}_2$, $T(f(x)) = f(x) - f'(x)$.

(c) $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $T(X) = AXB$ where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$.

(31) Let V be a linear space of dimension n and $\mathcal{B} = (f_1, \dots, f_n)$ a basis of V. For $f \in V$, what is the definition of the $\mathcal{B}$-coordinate vector $[f]_\mathcal{B}$? Make sure you understand how to translate linear algebra questions about V into questions about $\mathbb{R}^n$ using the $\mathcal{B}$-coordinate transformation $L\colon V \to \mathbb{R}^n$, $L(f) = [f]_\mathcal{B}$.

(32) Let $V = \mathcal{P}_2$ and $\mathcal{B} = (1, (x-1), (x-1)^2)$ a basis of V. Let $f = x^2 + 2x + 3 \in V$. Compute $[f]_\mathcal{B}$.

(33) Let V be a linear space of dimension n and $\mathcal{B} = (f_1, \dots, f_n)$ a basis of V. Let $T\colon V \to V$ be a linear transformation from V to V. What is the definition of the $\mathcal{B}$-matrix of T? Explain how to compute the columns of the $\mathcal{B}$-matrix.

(34) For parts (a) and (c) of Q28, write down a basis $\mathcal{B}$ of V and compute the $\mathcal{B}$-matrix of T.

(35) Let $C^\infty(\mathbb{R}, \mathbb{R})$ denote the linear space of functions $f\colon \mathbb{R} \to \mathbb{R}$ having derivatives of all orders. Let $V \subset C^\infty(\mathbb{R}, \mathbb{R})$ be the subspace spanned by $\cos(x)$ and $\sin(x)$.

(a) Explain why $\mathcal{B} = (\cos(x), \sin(x))$ is a basis of V.
(b) Let $T\colon V \to V$ be the linear transformation defined by $T(f(x)) = f'(x) + f(x)$. Compute the $\mathcal{B}$-matrix of T. Is T invertible?

(36) Let V be a linear space, $\mathcal{B}$ a basis of V, and $\mathcal{C}$ another basis of V. Recall that the change of basis matrix $S_{\mathcal{B}\to\mathcal{C}}$ is defined by

$$S_{\mathcal{B}\to\mathcal{C}} \cdot [\mathbf{x}]_\mathcal{B} = [\mathbf{x}]_\mathcal{C} \text{ for all } \mathbf{x} \in V.$$

Explain how to compute the columns of $S_{\mathcal{B}\to\mathcal{C}}$.

(37) Let $V \subset \mathbb{R}^3$ be the plane defined by the equation $x + 2y + z = 0$. Let $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ be the basis of V given by $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$.

Let $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2)$ be another basis of V given by $\mathbf{w}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(a) Compute the change of basis matrix $S_{\mathcal{B}\to\mathcal{C}}$.

(b) Using part (a) or otherwise, compute $S_{\mathcal{C}\to\mathcal{B}}$.

(38) What is the area of the image $T(S)$ of the unit square

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; 0 \leq x \leq 1,\ 0 \leq y \leq 1 \right\} \subset \mathbb{R}^2$$

under the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$? What is the area of the image $T(D)$ of the unit disk

$$D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; x^2 + y^2 \leq 1 \right\}$$

under the linear transformation T?

(39) Compute the determinant of each of the following matrices. Use your answer to determine whether the matrix is invertible.

(a) $A = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 2 & 1 & 6 & 7 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

(e) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 4 & 7 & 9 \\ 3 & 3 & 3 & 8 \end{pmatrix}$.

(40) Let $T\colon \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & x \\ 1 & 2 & y \\ 1 & 3 & z \end{pmatrix}.$$

(a) Explain why T is a linear transformation and compute the standard matrix of T.

(b) What is the kernel of T? Explain your answer geometrically.

(41) Let n be a positive integer. Let A and B be $n \times n$ matrices. Suppose $\det A = 5$ and $\det B = 7$. Compute (a) $\det(AB)$, (b) $\det(A^{-1})$ (note that A is invertible (why?)), (c) $\det(SAS^{-1})$ (where S is an invertible $n \times n$ matrix).

(42) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose that there is a basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of $\mathbb{R}^3$ such that $T(\mathbf{v}_1) = 3\mathbf{v}_1$, $T(\mathbf{v}_2) = 2\mathbf{v}_2$, and $T(\mathbf{v}_3) = 2\mathbf{v}_3 + \mathbf{v}_2$. What is the determinant of the standard matrix of T?