Linear Algebra Final Exam Practice Problems  
Math 235 Fall 2009

1: Let $V,W$ be vector spaces. Define the following terms:

1a: What is a subspace of $V$?

1b: Let $F : V \to W$ be a function. What does it mean to say that $F$ is linear?

1c: Let $T = \{v_1, v_2, \ldots\}$ be a subset of $V$. What is a linear combination of elements of $T$? What is the span of $T$? What does it mean to say that $T$ is linearly independent? What does it mean to say that $T$ spans $V$? What does it mean to say that $T$ is a basis of $V$?

1d: What is the dimension of $V$?

1e: Let $F : V \to W$ be linear. Define $\ker(F)$. Define $\text{im}(F)$. What is the rank of $F$? What is the nullity of $F$?

1f: Let $F : V \to V$ be linear. What is an eigenvalue of $F$? What is an eigenvector of $F$?

1g: What does it mean to say that two $n \times n$ matrices are similar?

1h: What does it mean to say that two vector spaces are isomorphic?

1i: Let $A$ be an $n \times n$ matrix. What is an eigenbasis for the matrix $A$?

1k: Let $B$ be a basis of a vector space $V$. What does one mean by the coordinates of a vector $v \in V$ with respect to $B$?

2a: Let $F : V \to W$ be linear. Show that $\ker(F)$ is a subspace of $V$. Show that $\text{im}(F)$ is subspace of $W$.

2b: State the rank+nullity theorem.

3: Consider the system of equations

$$
\begin{align*}
x - 2y + 3z - w &= 2 \\
2x + y - z + 3w &= 1 \\
5x + z + 5w &= 4.
\end{align*}
$$

3a: Find all, if any, solutions to this system.

3b: Write the system as a matrix equation.
4a: Which vectors \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) can be written as a linear combination of the vectors \( \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix} \).

4b: Which vectors \( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) are in the image of the matrix
\[
\begin{pmatrix}
1 & 0 & 2 \\
-2 & 1 & -5 \\
3 & -2 & 8
\end{pmatrix}.
\]

5: Let \( A \) denote the matrix representing rotation by angle \( \pi/6 \) about the line through the origin and the point \( \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \). Let \( B \) be the matrix representing reflection across the plane \( 3x - y + z = 0 \). How do you find the matrix representing the composition, first, of the reflection and then, second, the rotation from the matrices \( A \) and \( B \)? Note that we do not ask you to find the matrices \( A \) and \( B \) or the matrix representing the composition.

6: Let \( A = \{ \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \} \) be a basis of \( \mathbb{R}^2 \). Let
\[
M = \begin{pmatrix}
1 & -2 \\
3 & 0
\end{pmatrix}
\]
be the matrix representing a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) with respect to the basis \( E = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \). What is the representation this linear transformation with respect to the basis \( A \)?

7: True or False. (Explain!)

7a: The set of all vectors of the form \( \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \) where \( a, b \) are real numbers forms a subspace.

7b: Let \( V \) be the space of all functions from \( \mathbb{R} \) to \( \mathbb{R} \) that have infinitely many derivatives. The function
\[
F : V \rightarrow V \\
F : f \mapsto 3f' - 2f''
\]
is linear.

7c: If the determinant of a \( 4 \times 4 \) matrix is 4, then the rank of the matrix must be 4.
7d: If the standard vectors \( \{e_1, e_2, \ldots, e_n\} \) are eigenvectors of an \( n \times n \) matrix, then the matrix is diagonal.

7e: If 1 is the only eigenvalue of an \( n \times n \) matrix \( A \), then \( A \) must be \( I_n \).

7f: If two \( 3 \times 3 \) matrices both have the eigenvalues 3, 4, 5, then \( A \) must be similar to \( B \).

8a: Let \( F \) be counterclockwise rotation of the plane by angle 45 degrees followed by a scaling of \( \frac{3}{2} \). What are all the eigenvalues of \( F \).

8b: What are all the eigenvalues and eigenvectors of orthogonal projection onto a line \( L \) in \( \mathbb{R}^3 \)?

9: Let \( A \) be a \( 2 \times 2 \) matrix with eigenvalues 1.3, .6 and corresponding eigenvectors \( \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \). Let \( v = \begin{pmatrix} 15 \\ 15 \end{pmatrix} \). Find \( A^n(v) \) for \( n = 61 \). Your answer will have expressions of the form \( (.6)^p, (1.3)^p \). Do not simplify these.

10a: Find the eigenvalues and eigenvectors for the matrix

\[
A = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.
\]

Find a matrix \( B \) so that

\[
BAB^{-1} = M
\]

is diagonal. What is the matrix \( M \).

10b: Find the eigenvalues and eigenvectors for the matrix

\[
\begin{pmatrix} 8 & 9 \\ -4 & -4 \end{pmatrix}.
\]

Is this matrix diagonalizable. If it is what is the diagonal matrix? If not diagonalisable, why not?

10c: Find the eigenvalues and eigenvectors for the matrix

\[
C = \begin{pmatrix} 0 & 2 \\ -5 & 2 \end{pmatrix}.
\]

Give a matrix of the form \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \) that is similar to \( C \).
11: Find the eigenvalues of the matrix $A$, given below. Find bases for the eigenspaces of $A$. Can you find an invertible matrix, $S$, such that $S^{-1}AS = D$, where $D$ is a diagonal matrix? If no, why not? If yes, find the matrices $S$ and $D$.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 2 \end{pmatrix}.$$  

*Hint:* When computing the characteristic polynomial of $A$, watch out for common factors: you want it factored at the end of the day.

12: Find the eigenvalues of the matrix $A$, given below. Find bases for the eigenspaces of $A$. Can you find an invertible matrix, $S$, such that $S^{-1}AS = D$, where $D$ is a diagonal matrix? If no, why not? If yes, find the matrices $S$ and $D$.

$$A = \begin{pmatrix} -8 & 5 & 4 \\ -9 & 5 & 5 \\ 0 & 1 & 0 \end{pmatrix}.$$  

*Hint:* One way to solve a cubic equation is to find (guess) one root, and then perform long division, which would leave you with a quadratic polynomial. If the characteristic polynomial has a free coefficient which is an integer, as a first guess you may want to check the numbers which divide it. For example, if you have $\lambda^3 - 2\lambda^2 - \lambda + 2$, you may want to try $\pm1$ and $\pm2$.

13: Find the eigenvalues of the matrix $A$, given below. Find bases for the eigenspaces of $A$. Can you find an invertible matrix, $S$, such that $S^{-1}AS = D$, where $D$ is a diagonal matrix? If no, why not? If yes, find the matrices $S$ and $D$.

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 6 & 6 & -5 \end{pmatrix}.$$  

14: Find the determinant of the matrix

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 5 \\ 4 & -1 & 3 \end{pmatrix}.$$