Math 235 Fall 09 Practice Final Problems Answers

1 a. A subset S of the vector space V is a subspace if (a) $x, y \in S \Rightarrow x+y \in S$ and (b) $x \in S$, $\lambda \in \mathbb{R} \Rightarrow \lambda x \in S$.

b. $F: V \longrightarrow W$ is linear if $F(u+v) = F(u) + F(v)$, $F(\lambda u) = \lambda F(u)$
$u, v \in V$, $\lambda \in \mathbb{R}$.

c. $T = \{v_1 \dots\} \subseteq V$. A linear combination of elts of T is any expression of the form $\sum a_i v_i$ $a_i \in \mathbb{R}$, $v_i \in T$.

The span of T is set of all linear combinations of elts of T.

T spans V if the span of T is all of V.

T is a basis of V provided (i) T is independent and ii T spans V.

d. The dimension of V is the number of elts in a basis of V.

e. $F: V \to W$. $\ker(F) = \{v \in V \mid F(v) = 0\}$
$\operatorname{im}(F) = \{w \in W \mid w = F(v),$ some $v \in V\}$

$\operatorname{rank}(F) = \dim(\operatorname{im}(F))$. Nullity of $F = \dim(\ker(F))$.

f. $F: V \to V$. An eigenvector of F is an elt $v \in V$ so that $F(v) = \lambda v$
$\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$ we must have $v \neq 0$. ~~We must have~~ The number λ is the eigenvalue of F associated to v.

g: A, B are similar if $\exists\ X$ so $A = XBX^{-1}$.

h: Two vectorspaces V, W are isomorphic if there exists $f: V \to W$ that is linear and has an inverse.

i. An eigenbasis for A is a basis B of $\mathbb{R}^n$ so that $Av_i = \lambda_i v_i$ for some λ_i for all ~~[illegible]~~ $v_i \in B$.

k. B = basis of V. $v \in V$. We can write $v = \sum \lambda_i b_i$ $\lambda_i \in \mathbb{R}$, $b_i \in B$.
Then ~~[illegible]~~ $(\lambda_1, \lambda_2, \lambda_3, \dots)$ are coordinates of v with respect to B.

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2 a. $F: V \to W$. Assertion: $\ker(F)$ is a subspace

Proof: (i). Let $u, v \in \ker F$. Then $F(u+v) = F(u) + F(v)$ (F is linear)
$= 0 + 0 = 0$.

(ii) Let $u \in \ker F$, $\lambda \in \mathbb{R}$. Then $F(\lambda u) = \lambda F(u) = \lambda \cdot 0 = 0$. ∎
(F is linear)

Assertion $\operatorname{im}(F)$ is a subspace. Proof: (i) Let $w \in \operatorname{im} F$, so $w = F(u)$ and let $x \in \operatorname{im} F$ so $x = F(v)$ $(u, v \in V)$. Then $F(u+v) = F(u) + F(v)$
$= w + x$. $\therefore$ $w + x \in \operatorname{im}(F)$.

(ii) Let $w \in \operatorname{im} F$, $\lambda \in \mathbb{R}$. So $w = F(u)$

Then $F(\lambda u) = \lambda F(u) = \lambda \cdot w$. $\therefore$ $\lambda w \in \operatorname{im}(F)$. ∎

(b). Let $F: V \to W$. If $\dim V$ is finite, then $\dim V = \operatorname{rank}(F) + \operatorname{nullity}(F)$.
F linear.

3(a)
$$\begin{aligned} x - 2y + 3z - w &= 2 \\ 2x + y - z + 3w &= 1 \\ 5x + z + 5w &= 4 \end{aligned} \longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 2 & 1 & -1 & 3 & 1 \\ 5 & 0 & 1 & 5 & 4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 0 & 5 & -7 & 5 & -3 \\ 0 & 10 & -14 & 10 & -6 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 0 & 5 & -7 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1/15 & 1 & 4/5 \\ 0 & 5 & -7 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x = -\frac{1}{5} \cdot z - w + 4/5$

$y = +\frac{7}{5} \cdot z - w - 3/5$

z = anything

w = anything

3b.
$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & 1 & -1 & 3 \\ 5 & 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

4a: $x\begin{pmatrix}1\\-2\\3\end{pmatrix}+y\begin{pmatrix}0\\1\\-2\end{pmatrix}+z\begin{pmatrix}2\\-5\\8\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix} \longrightarrow \left(\begin{array}{ccc|c}1&0&2&a\\-2&1&-5&b\\3&-2&8&c\end{array}\right)$

$\longrightarrow \left(\begin{array}{ccc|c}1&0&2&a\\0&1&-1&b+2a\\0&-2&2&c-3a\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1&0&2&a\\0&1&-1&b+2a\\0&0&0&a+2b+c\end{array}\right)$. This has a solution $\iff$ $a+2b+c=0$.

4b: $\begin{pmatrix}1&0&2\\-2&1&-5\\3&-2&8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix} \iff x\begin{pmatrix}1\\-2\\3\end{pmatrix}+y\begin{pmatrix}0\\1\\-2\end{pmatrix}+z\begin{pmatrix}2\\-5\\8\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ $\therefore$

We get from 4a: $\begin{pmatrix}a\\b\\c\end{pmatrix} \in$ image $\iff$ $a+2b+c=0$.

5. AB $\Leftarrow$ multiply the two matrices in this order.

6. $I_{E\leftarrow A} = \begin{pmatrix}1&1\\2&3\end{pmatrix}$ $\qquad I_{A\leftarrow E} = \begin{pmatrix}1&1\\2&3\end{pmatrix}^{-1} = \begin{pmatrix}3&-1\\-2&1\end{pmatrix}$.

$\begin{pmatrix}3&-1\\-2&1\end{pmatrix}\begin{pmatrix}1&-2\\3&0\end{pmatrix}\begin{pmatrix}1&1\\2&3\end{pmatrix}$ = rep of matrix WRT basis A. You should multiply this set of matrices.

$A\leftarrow E \quad E\leftarrow E \quad E\leftarrow A$

7.(a) True. Why: $\begin{pmatrix}a_1\\b_1\\0\\a_1\end{pmatrix}+\begin{pmatrix}a_2\\b_2\\0\\a_2\end{pmatrix}=\begin{pmatrix}a_1+a_2\\b_1+b_2\\0\\a_1+a_2\end{pmatrix}$ is of same form.

$\lambda\begin{pmatrix}a\\b\\0\\a\end{pmatrix}=\begin{pmatrix}\lambda a\\\lambda b\\0\\\lambda a\end{pmatrix}$ is of same form as elts in set.

7b. True: (i) $F(f+g) = 3\left((f+g)' - 2(f+g)''\right)$ $= 3(f'+g') - 2(f''+g'') = 3f'-2f''+3g'-2g'' = F(f)+F(g)$

ii) $F(\lambda f) = 3(\lambda f)' - 2(\lambda f)'' = 3\lambda f' - 2\lambda f''$ $= \lambda F(f)$

7c. True: $\det \neq 0 \Rightarrow$ invertible $\Rightarrow$ rank $=4$.

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7d: True. Let A = $n \times n$ matrix. Note: $Ae_i = i^{th}$ col vector of A.
Assume $Ae_i = \lambda e_i$. This says that when we construct A it is diagonal with entries $\lambda_1, \lambda_2, \dots \lambda_n$

7e: False $\begin{pmatrix}1&1\\0&1\end{pmatrix}$

7f: True. If A is 3×3 with eigenvalues we can find a matrix X so that $X^{-1}AX = \begin{pmatrix}3&&\\&4&\\0&&5\end{pmatrix}$. Similarly we can find Y so that $Y^{-1}BY = \begin{pmatrix}3&&\\&4&\\0&&5\end{pmatrix}$. So $X^{-1}AX = Y^{-1}BY \Rightarrow$

$$XY^{-1}BYX^{-1} = (XY^{-1})(B)(XY)^{-1}.$$

8 (a): $\frac{3}{2}\left(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right)$

(b). Eigenvalues are 0, 0 and 1. Let f_1, f_2 be $\perp$ to L and let f_3 be in L. Then these ~~...~~ are eigenvectors with eigenvalues 0, 0, 1.

9: $\begin{pmatrix}1&1\\2&3\end{pmatrix} = X = I_{E\leftarrow F}$ $\left(E = \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}.\ F = \left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}\right\} = \text{basis of } \mathbb{R}^2.\right)$

Then $X^{-1}AX = \begin{pmatrix}1.3&0\\0&.6\end{pmatrix}$ ∴ $A^n = X\begin{pmatrix}1.3&0\\0&.6\end{pmatrix}^n X^{-1}$

$= \begin{pmatrix}1&1\\2&3\end{pmatrix}\begin{pmatrix}1.3^n&0\\0&.6^n\end{pmatrix}\begin{pmatrix}3&-1\\-2&1\end{pmatrix}$. Now mult this times $\begin{pmatrix}15\\15\end{pmatrix}$. Rest is left to student.

10a: i) $\det\begin{pmatrix}4-\lambda&2\\2&7-\lambda\end{pmatrix} = \lambda^2 - 11\lambda + 24 = (\lambda-3)(\lambda-8)$ ∴ eigenvalues are $\lambda = 3, \lambda = 8$.

ii). eigenvectors: $(A-3I)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \iff \begin{pmatrix}1&2\\2&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

One solution to this is $\begin{pmatrix}2\\-1\end{pmatrix}$. This is eigenvector with eigenvalue $\lambda = 3$.

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10a (cont'd.) — eigenvectors — $\lambda = 8$.

$$A - 8I = \begin{pmatrix} \cancel{} -4 & 2 \\ 2 & -1 \end{pmatrix}. \qquad \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ has solution}$$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This is eigenvector for $\lambda = 8$.

With respect to the basis $\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ the linear map is represented by matrix $M = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$. $M = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}(A)\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}^{-1}$.

10b. $\det\begin{pmatrix} 8-\lambda & 9 \\ -4 & -4-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$. $\therefore$ $\lambda = 2$ is only eigenvalue. Eigenvectors: $\begin{pmatrix} 8-2 & 9 \\ -4 & -4-2 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}$

This has kernel spanned by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. $\therefore$ Not diagonalizable since $\mathbb{R}^2$ does not have eigenbasis for matrix $\begin{pmatrix} 8 & 9 \\ -4 & -4 \end{pmatrix}$.

10c: $\det\begin{pmatrix} -\lambda & 2 \\ -5 & 2-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 10$. This has roots

$$\lambda = \frac{2 \pm \sqrt{4 - 40}}{2} = \cancel{4 \pm \sqrt{3}i}.\ 1 \pm \sqrt{3}i$$

eigenvectors for $\lambda = 1 + \sqrt{3}i$: $\begin{pmatrix} -1-\sqrt{3}i & 2 \\ -5 & 2-1-\sqrt{3}i \end{pmatrix} \rightarrow$

$\begin{pmatrix} -1-\sqrt{3}i & 2 \\ -5 & 1-\sqrt{3}i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\sqrt{3}i \end{pmatrix}$.

Similarly for $1 - \sqrt{3}i$. The matrix $\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ is similar to C.

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11. $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix}$

$= (\lambda^2-2\lambda-3)(\lambda^2-4\lambda-5) = (\lambda-3)(\lambda+1)(\lambda-5)(\lambda+1)$

$\therefore$ The eigenvalues are $\lambda=3$, $\lambda=-1$, $\lambda=5$, $\lambda=-1$.

We find the eigenvectors for $\lambda=-1$. This means we have to find a basis for $\ker(A+I)$. But $A+I = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}$. We use row reduction. We get $\begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$ $x_1 = -x_2$, $x_2 = x_2$, $x_3 = -x_4$, $x_4 = x_4$

$\therefore$ get a basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ for $\ker(A+I)$. This is a basis for the eigenspace for the eigen~~vector~~value $\lambda=-1$.

An eigenvector for $\lambda=3$ is $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. $\left[\ker(A-3I) = \ker\begin{pmatrix} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 3 & -1 \end{pmatrix} \right]$

An eigenvector for $\lambda=5$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Then $1_{F\leftarrow E}\, A\, 1_{E\leftarrow F} =$

Let $S = 1_{E\leftarrow F} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

$S^{-1}AS = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

12. $\det(A-\lambda I) = \det\begin{pmatrix} -8-\lambda & 5 & 4 \\ -9 & 5-\lambda & 5 \\ 0 & 1 & -\lambda \end{pmatrix} = -1\det\begin{pmatrix} -8-\lambda & 4 \\ -9 & 5 \end{pmatrix} - \lambda\det\begin{pmatrix} -8-\lambda & 5 \\ -9 & 5-\lambda \end{pmatrix}$

$= -\lambda^3 - 3\lambda^2 + 4 = -(\lambda^3+3\lambda^2-4)$. We try $\lambda=\pm1$, $\lambda=\pm2$, $\lambda=\pm4$ as roots. We get $\lambda=1$ is a root. Use long division:

$$\begin{array}{r} \lambda^2+4\lambda+4 \\ \lambda-1 \enclose{longdiv}{\lambda^3+3\lambda^2-4} \\ \underline{\lambda^3-\lambda^2} \\ 4\lambda^2 \\ 4\lambda^2-4\lambda \\ 4\lambda-4 \end{array}$$

$\therefore \lambda^3+3\lambda^2-4 = (\lambda-1)(\lambda^2+4\lambda+4)$

$\therefore$ eigenvalues $\lambda=1$, $\lambda=-2$, $\lambda=-2$.

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12. Cont'd. We find the eigenvalues for $\lambda = -2$. This means we find a basis for $\ker(A-\lambda I) = \ker(A+2I) = \ker\left(\begin{pmatrix}-8 & 5 & 4\\ -9 & 5 & 5\\ 0 & 1 & 0\end{pmatrix} + \begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}\right)$

$= \ker\begin{pmatrix}-6 & 5 & 4\\ -9 & 7 & 5\\ 0 & 1 & 2\end{pmatrix}$. We use row reduction:

$$\begin{pmatrix}-6 & 5 & 4\\ -9 & 7 & 5\\ 0 & 1 & 2\end{pmatrix} \to \begin{pmatrix}1 & -5/6 & -2/3\\ -9 & 7 & 5\\ 0 & 1 & 2\end{pmatrix} \to \begin{pmatrix}1 & -5/6 & -2/3\\ 0 & -1/2 & -1\\ 0 & 1 & 2\end{pmatrix} \to \begin{pmatrix}1 & -5/6 & -2/3\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$$\to \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$\therefore$ $x_1 = -x_3$ describes kernel. Let $x_3 = 1$

$x_2 = -2x_3$

$x_3 = x_3$ get $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ as basis of eigenspace for $\lambda = -2$.

Since multiplicity of $\lambda = -2$ was 2 and the eigenspace has dim 1 we conclude that this matrix is NOT diagonalizable.

13. Let $A = \begin{pmatrix}3 & 2 & -2\\ 2 & 3 & -2\\ 6 & 6 & -5\end{pmatrix}$ $\chi_A(\lambda) = \det(A-\lambda I) = 3\det\begin{pmatrix}3-\lambda & -2\\ 6 & -5-\lambda\end{pmatrix}$

$-2\det\begin{pmatrix}2 & -2\\ 6 & -5-\lambda\end{pmatrix} + (-2)\det\begin{pmatrix}2 & 3-\lambda\\ 6 & 6\end{pmatrix} = -\lambda^3 + \lambda^2 + \lambda - 1 = \chi_A(\lambda)$

Possible integer roots are $\lambda = \pm 1$. Plug $\lambda = +1$ into $\chi_A(\lambda)$. You get zero. Plug in $\lambda = -1$ into $\chi_A(\lambda)$, you get zero. $\therefore$ $(\lambda-1)(\lambda+1) = \lambda^2 - 1$ are both factors.

$$\begin{array}{r} \lambda - 1 \\ \lambda^2 - 1 \overline{\smash{)}\ \lambda^3 - \lambda^2 - \lambda + 1} \\ \underline{\lambda^3 - \lambda } \\ -\lambda^2 + 1 \end{array}$$

$\therefore$ $\chi_A(\lambda) = -(\lambda-1)^2(\lambda+1)$.

We find that the eigenspace for $\lambda = +1$ is spanned by $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$. The vector $\begin{pmatrix}1\\1\\3\end{pmatrix}$ spans the eigenspace for $\lambda = -1$.

$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\3\end{pmatrix}\right\}$ is an eigenbasis for A. A is similar to $D = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$

Indeed $D = SAS^{-1}$ with $S = \begin{pmatrix}1 & -1 & 1\\ 0 & 1 & 1\\ 1 & 0 & 3\end{pmatrix}$.

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14. (a) $\det\begin{pmatrix}-1 & 2 & 0\\ 2 & -2 & 5\\ 4 & -1 & 3\end{pmatrix} = -1\det\begin{pmatrix}-2 & 5\\ -1 & 3\end{pmatrix} - 2\det\begin{pmatrix}2 & 5\\ 4 & 3\end{pmatrix}$

$= (-1)(-1) - 2(-14) = 29.$

(b) $\det\begin{pmatrix}-1 & 2 & 0\\ 2 & -2 & 5\\ 4 & -1 & 3\end{pmatrix} = -\det\begin{pmatrix}1 & -2 & 0\\ 2 & -2 & 5\\ 4 & -1 & 3\end{pmatrix} = \cancel{-\det\begin{pmatrix}1 & 2 & 0\\ 0 & -6 & 5\\ 0 & -9 & 3\end{pmatrix}}$

$= -\det\begin{pmatrix}1 & -2 & 0\\ 0 & 2 & 5\\ 0 & 7 & 3\end{pmatrix} = (-1)(6-35) = 29.$

Both methods are important.