Practice Problems Midterm Exam November 12 7pm

1: Let V, W be a vector spaces and let $F : V \longrightarrow W$ be a linear transformation of vector spaces.
1a: Define ker(F), im(F).
1c: Let V be a vector space and let $S \subset V$ be a subset of V. Define what it means for S to be a subspace of V.
1d: let V be a vector space. Define what the dimension of V is.
1d: Define rank and nullity of F.

2: Let $F : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be given by the matrix

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 & 1 \\ 7 & -1 & 3 & -5 & 5 \end{pmatrix}.$$

Find a basis of $ker(F)$ and $im(F)$. What is the rank of F?
3: Let $A = \{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$ be a basis of $\mathbb{R}^2$. Let E be the basis $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

3a: For a vector $v \in \mathbb{R}^2$and let v_A denote its expression in terms of A coordinates and let v_E denote its expression in terms of E coordinates. Find a matrix $1_{A \leftarrow E}$ so that

$$1_{A \leftarrow E} v_E = v_A.$$

3b: Find a matrix $1_{E \leftarrow A}$ so that

$$1_{E \leftarrow A} v_A = v_E.$$

3c: If $v_E = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then what is v_A?
3d: Let $M = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ be a matrix representing a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ with respect to the basis E. What is the matrix of this linear transformation with respect to the basis A?

4: Let P be the plane in $\mathbb{R}^3$ defined by $2x - y + z = 0.$. Let R be reflection across P. Find a basis B of $\mathbb{R}^3$ so that the matrix of R with respect to the basis B is diagonal.

5: True or False. You must explain.
5a: If $v_1, v_2, \cdots v_n$ are independent vectors in $\mathbb{R}^n$, then they form a basis of $\mathbb{R}^n$.
5b: There is a 5×4 matrix whose image is all of $\mathbb{R}^5$.
5c: If

$$2u + 3v + 4w = 3u - 4v + 5w,$$

then the vectors u, v, w are dependent.
5d: The column vectors of a 5×4 matrix must be dependent.

6a: Let P_2 denote the vector space of all polynomials of degree less than or equal to 2 in the variable t. Is

$$S = \{p \in P_2 | \frac{dp}{dt}(1) = p(2)\}$$

a subspace of P_2. Explain why.
6b: Is the set of all 3×3 matrices which are invertible a subsapce of $\mathbb{R}^{3\times 3}$? Explain why.
6c: Let $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Is the set S of all matrices $A \in \mathbb{R}^{3\times 3}$ such that $Av = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ a subspace of $\mathbb{R}^{3\times 3}$? Explain why.
6d: In problems 6a, 6b if the set S we describe is a subspace, then find a basis for it.

7: Find a basis of the following vector spaces. What is the dimension of each?
7a: $\mathbb{R}^{n\times m}$.
7b: P_n, the space of polynomials of degree less than n.
7c: $\mathbb{C}^2$ considered as a vector space over $\mathbb{R}$.
7d: Let $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The space of all 2×2 matrices A such that $AB = BA$.

8: In parts a-e of this problem detemine which of the following are linear maps. If the map is linear, then determine whether it is an isomorphism or not.
8a: Define T by

$$T : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2},$$

with

$$M \mapsto M^2.$$

8b: $T : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ with

$$T : M \mapsto \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} M.$$

8c: $T : \mathbb{C} \longrightarrow \mathbb{C}$ with

$$T : x + iy \mapsto x.$$

8d: $T : \mathbb{C} \longrightarrow \mathbb{C}$ with

$$T : x + iy \mapsto (3 - 4i)(x + iy).$$

8e: Let P_2 denote the vector space of polynomials of degree less than or equal to to 2. Let $T : P_2 \longrightarrow P_2$ map

$$f \mapsto f'' - 3f'.$$

8f: For the transformation T in problem 8e determine a basis of $ker(T)$.
8g: For the transformation in problem 8b determine the rank and nullity.
9: We set up some notation.
Let C be the basis $\{1, i\}$ of $\mathbb{C}$.
Let A be the basis $\{1, t, t^2\}$ of P_2, the vector space of all polynomials of degree less than or equal to 2 in the variable t.
Let B be the basis

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of U, the vector space of upper triangular 2×2 matrices.

Find the matrix of the linear transformations below with respect to the indicated basis. Determine if the linear map is an isomorphism. If it is not an isomorphism find a basis of the kernel and the image.
9a: Let $T : U \longrightarrow U$ map

$$M \mapsto M \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M.$$

Let B be the basis of both the domain and target (also called the range) of T.
9b: Let $F : \mathbb{C} \longrightarrow \mathbb{C}$ send

$$x + iy \mapsto x - iy.$$

Let C be the basis of both the domain and the target (also called the range).
9c: Let $G : \mathbb{C} \longrightarrow \mathbb{C}$ map

$$x + iy \mapsto (4 - 6i)(x + iy).$$

Let C be the basis of both domain and target (also called the range).
9d: Let $D : P_2 \longrightarrow P_2$ map

$$f \mapsto f' + 3f.$$

Let A be the basis of both domain and target (also called the range).
9e: Let $E : P_2 \longrightarrow \mathbb{R}$ be the map sending

$$f \longrightarrow f(3).$$

Let A be the basis of the domain and $\{1\}$ be the basis of $\mathbb{R}$.

10: Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ with infinitely many derivatives.
10a: Consider the map $F : V \longrightarrow V$ given by

$$f(x) \mapsto f(3x) + 2f(x).$$

Is F linear? Jusify your answer.
10b: Is the map $V \longrightarrow V$ given by

$$f(x) \mapsto f'(x) + 5f(x)$$

linear? Justify your answer.

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