1(a) Let $F: V \to W$ be linear. $\ker(F) = \{v \in V \mid F(v) = 0\}$.

$\operatorname{im}(F) = \{w \in W \mid$ there exists $x \in V$ so that $F(x) = w\}$.

(c). $S \subseteq V$ is a subspace $\iff$ (1) $x, y \in S \Rightarrow x + y \in S$, and

(2) $x \in S, \lambda \in \mathbb{R} \Rightarrow \lambda x \in S$.

(d). dim V equals the number of elements in a basis of V. If there is no finite basis of V we say the dimension is infinite.

2: $\begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 & 1 \\ 7 & -1 & 3 & -5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & 6 & -4 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2/3 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/3 & -2/3 & 2/3 \\ 0 & 1 & -2/3 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

$x_1 = -1/3\, x_3 + 2/3\, x_4 - 2/3\, x_5$
$x_2 = 2/3\, x_3 - 1/3\, x_4 + 1/3\, x_5$
$x_3 = x_3$
$x_4 = x_4$
$x_5 = x_5$

$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Conclude. $\left\{ \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of ker F

and $\left\{ \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis of im(F). The rank of F is 2

3. If $A = \left\{ \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{f_1}, \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{f_2} \right\}$. Then $f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \rightsquigarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}_E$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_B \longrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}_E$.

$\therefore 1_{E \leftarrow A} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ and (This is 3b)

$1_{A \leftarrow E} = (-1)\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$. (This is 3a)

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3c. $v_E = \begin{pmatrix}1\\-2\end{pmatrix} \Rightarrow v_A = I_{A\leftarrow E}\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}-2&1\\3&-1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}_E = \begin{pmatrix}-4\\5\end{pmatrix}_A$

3d. $M_{A\leftarrow A} = \left(I_{A\leftarrow E}\right)\begin{pmatrix}1&-1\\2&3\end{pmatrix}\left(I_{E\leftarrow A}\right) = \begin{pmatrix}-2&1\\3&-1\end{pmatrix}\begin{pmatrix}1&-1\\2&3\end{pmatrix}\begin{pmatrix}1&1\\3&2\end{pmatrix} =$ L.T.S.

4. $2x - y + z = 0$ is the plane thru the origin $\perp$ to $(2,-1,1)$. We find a basis of this plane. That is, we find a basis of the kernel of linear map associated to matrix $M = (2\ -1\ 1)$. Setting

$y = 1, z = 0$ we get $x = \frac{1}{2}$ so $\begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix} \in$ kernel of M

Set $y = 0, z = 1$ we get $x = -\frac{1}{2}$ so $\begin{pmatrix}-\frac{1}{2}\\0\\1\end{pmatrix} \in$ kernel of M.

Since these two vectors ~~are~~ are in P we have $R\begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix}$

$R\begin{pmatrix}-\frac{1}{2}\\0\\1\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\0\\1\end{pmatrix}$. We have $R\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$. Let these three vectors be our basis. Then the matrix is $\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ (suitably order these vectors).

5(a). True. [[Let M be the matrix with columns $v_1, \ldots v_n$. Then $\ker(M) = 0 \Rightarrow$ (by rank-nullity) rank $= n = \dim \operatorname{im}(M)$

But $\mathbb{R}^n$ is only subspace of $\mathbb{R}^n$ with dim n. $\therefore \operatorname{im}(M) = \mathbb{R}^n$. This says span of $v_1 \ldots v_n$ is all of $\mathbb{R}^n$]] OR [[(better). Let W be subspace spanned by $\{v_1, \ldots v_n\}$. It is dim $\geq n$; thus it has to $= \mathbb{R}^n$]]

(b) False: The image is spanned by the column vectors. Thus any basis of the image has ≤ 4 elts, but $\dim \mathbb{R}^5 = 5$.

(c) True: $2u + 3v + 4w = 3u - 4v + 5w \Rightarrow$

$0 = u - 7v + w$.

This is a non-trivial linear relation among $\{u, v, w\}$

Thus this set of vectors is linearly dependent

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5d. False: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

6 (a) It is a subspace. (a) Let $f, g \in S$, so $\frac{df}{dt}(1) = f(2)$ and $\frac{dg}{dt}(1) = g(2)$. Then $\frac{d}{dt}(f+g)(1) = \frac{df}{dt}(1) + \frac{dg}{dt}(1)$

$\overset{\text{hypothesis}}{=} f(2) + g(2) = (f+g)(2)$.

(b) Let $f \in S$, $\lambda \in \mathbb{R}$, so $\frac{df}{dt}(1) = f(2)$.

Then $\frac{d}{dt}(\lambda f)(1) = \lambda \frac{df}{dt}(1) \overset{\text{hypothesis}}{=} \lambda f(2) = (\lambda f)(2)$.

$\therefore$ yes S is a subspace.

(b) The set of all 3×3 invertible matrices is NOT a subspace.

For example $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) Let $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. The set of 3×3 matrices A so that $Av = 0$ is a subspace of $\mathbb{R}^{3 \times 3}$: (a) Assume $Av = 0$, $Bv = 0$. Then $(A+B)(v) = Av + Bv = 0 + 0 = 0$.

(b) Assume $Av = 0$, then $(\lambda A)v = \lambda(Av) = \lambda \cdot 0 = 0$

(d) In 6a find a basis of S. Let $f = at^2 + bt + c$. ~~Then~~ Then

~~$f(1) = a+b+c$ $\quad \frac{df}{dt} = 2at + b$ $\quad \frac{df}{dt}\Big|_{t=2} = 4a + b$~~

~~Then $f \in S \iff a + b + c = 0$, $4a + b = 0$. Let $M = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \end{pmatrix}$.~~

~~Find basis of ker M.~~ $\therefore \frac{df}{dt} = 2at + b$ $\quad \frac{df}{dt}\Big|_{t=1} = 2a + b$

$f(2) = 4a + 2b + c$ $\therefore$ We are looking for a, b, c so that $4a + 2b + c = 2a + b$ $\therefore$ looking for solutions to $2a + b + c = 0$.
This is kernel of matrix $(2\ 1\ 1)$. LTS.

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7a. $\mathbb{R}^{m\times n}$ has dim $m\cdot n$. A basis consists of the matrices with all zero entries ~~and~~ except 1 position. In that position put a 1. There are $m\times n$ positions.

7b. P_n has dim $n+1$. A basis is $\{1, t, \dots t^n\}$.

7c. $\mathbb{C}^2$ has dim 4. A basis is $\{(1,0), (i,0), (0,1), (0,i)\}$

7d. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$

$$BA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}.$$

$AB = BA \iff c = 0$ and $a = d$. ~~$c=0$~~.

Thus A commutes with $B \Rightarrow$ it is of the form

$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. ~~This has basis~~ The set of such matrices has a basis $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$. Thus dimension is 2.

8. 8a. $T: M \to M^2$ is not linear since $T(\lambda M) = \lambda^2 T(M)$

8b. $M \to \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} M$ is linear. It is an isomorphism. It has an inverse:

$M \to \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \cdot M$. You Need to say ~~that~~ why $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has an inverse. For example it has two leading ones when row reduced. OR the det is $\neq 0$.

8c. $T: x+iy \longrightarrow x$. This is linear. It is not an isomorphism. For example $T(2i) = 0$.

8d. $T: x+iy \longrightarrow (3+4i)(x+iy)$ is an isomorphism. It has an inverse: $\frac{1}{5}(3-4i)$.

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8e. $T: f \longrightarrow f'' - 3f'$ $f \in P_2$. is linear. It is not an isomorphism since $T(1) = 0$.

8f : $T: at^2 + bt + c \longrightarrow (2a) - 3(2at + b) = -6at + (2a - 3b) = 0$

$\iff a = 0$ and $2a - 3b = 0 \implies$ ~~a = b~~ $a = 0$ and $b = 0$.

$\therefore \ker T = \{ c \cdot 1 \mid c \in \mathbb{R} \}$ $\therefore$ A basis for $\ker T$ is the function 1.

8g: $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is an isomorphism.
$\therefore$ nullity $= 0$ and rank $= 4$.

9a: $T: M \longrightarrow M \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -2a+b \\ 0 & c \end{pmatrix} - \begin{pmatrix} a & b+c \\ 0 & c \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2a+2b+c \\ 0 & 0 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\therefore$ matrix of T is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}$. This is not an isomorphism.

(Use row reduction on matrix of T) $\to \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\therefore$ kernel in basis B has a basis

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}_B, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}_B \right\} \longrightarrow \left\{ \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

The image is spanned by $\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}_B \longrightarrow \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$.

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9b. $T: x+iy \longrightarrow x-iy$

$1 \longrightarrow 1$

$i \longrightarrow -i$

in terms of basis $C=\{1,i\}$ this has matrix $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. It is an isomorphism.

9c: $x+iy \longrightarrow (4-6i)(x+iy)$

$1 \longmapsto 4-6i$

$i \longmapsto 6+4i$

or in terms of basis C $\begin{pmatrix}1\\0\end{pmatrix}_C \to \begin{pmatrix}4\\-6\end{pmatrix}_C$, $\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}6\\4\end{pmatrix}$.

$\therefore T \longleftrightarrow \begin{pmatrix}4&6\\-6&4\end{pmatrix}_C$. This is an isomorphism.

9d: $T: P_2 \to P_2$
$f \longmapsto f'+3f$

$1 \to 0+3\cdot 1$

$t \to 1+3t$

$t^2 \longrightarrow 2t+3t^2$

$\Longleftrightarrow$ $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}3\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\3\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\2\\3\end{pmatrix}$

$\therefore$ T corresponds to the matrix $\begin{pmatrix}3&1&0\\0&3&2\\0&0&3\end{pmatrix}_{A\leftarrow A}$.

It has 3 leading ones when row reduced; hence it is an isomorphism

9e.: $T: P_2 \to \mathbb{R}$.
$f \longrightarrow f(3)$

$1 \longmapsto 1$

$t \longmapsto 3$

$t^2 \longrightarrow 9$

$\longleftrightarrow$ $\begin{pmatrix}1\\0\\0\end{pmatrix} \to 1$, $\begin{pmatrix}0\\1\\0\end{pmatrix} \to 3$, $\begin{pmatrix}0\\0\\1\end{pmatrix} \to 9$

$\therefore \begin{pmatrix}1&3&9\end{pmatrix} \Leftrightarrow T$ using our bases.

10. $F: V \to V$
$f \longmapsto f(3x)+2f(x)$.

Is F linear? (a) $(F(f+g))(x) =$

$(f+g)(3x) + 2(f+g)(x) = f(3x)+2f(x) + g(3x) + 2g(x) = Ff + Fg$.

(b) $(F(\lambda f))(x) = (\lambda f)(3x) + 3(\lambda f)(x)$

$= \lambda f(3x) + 3(\lambda f(x)) = \lambda(f(3x)+3f(x)) = \lambda(F(f))(x)$

Yes. It is linear.

10b. Let $G: V \to V$ be given by $f(x) \to f'(x) + 5f(x)$.

This is linear, (a) $G(f+g)(x) = (f+g)'(x) + 5(f+g)(x)$

$$= f'(x) + g'(x) + 5(f(x) + g(x)) = G(f)(x) + G(g)(x).$$

(b). $G(\lambda f)(x) = (\lambda f)'(x) + (5(\lambda f))(x)$

$$= \lambda f'(x) + 5 \cdot \lambda f(x) = \lambda (Gf)(x).$$ Done.