Final Exam Math 235 Fall 2008

Name:

1. a: Solve the system of equations using row operations.

$$x + y = z = 6$$

$$2x - y = 0$$

$$3x - y - 2z = -3$$

   b: Write the above system of equations as a matrix equation.

   c: Compute the matrix products AB and BA if possible:

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -5 & 3 \end{pmatrix}$$

   and

$$B = \begin{pmatrix} -1 & -1 & 2 \\ -2 & -3 & -2 \end{pmatrix}.$$

   d: For what vectors $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ does the equation $Ax = v$ have a solution if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

2. a: What does it mean for a basis of $\mathbb{R}^3$ to be orthonormal.
   b: Let $f_1 = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, $f_2 = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $f_3 = (-1/\sqrt{6}, 1/\sqrt{6}, \sqrt{2/3})$ .This is an orthonormal basis of $\mathbb{R}^3$. Let $v = (1, 1, 1)$. We can write $v = a_1 f_1 + a_2 f_2 + a_3 f_3$ with $a_1, a_2, a_3 \in \mathbb{R}$. Find a_1 by using the fact that the basis $\{f_1, f_2, f_3\}$ is orthonormal.

3. a: Define what it means for a subset of a vector space to be a basis of that subspace.

   b: Is $\{1, (t-1), (t-1)^2, (t-1)^3\}$ a basis of P_3. Here P_3 denotes the vector space of all polynomials of degree less than or equal to 3. Why? Again note that explaining why is the important part of the question.

4. a: Let V, W be vector spaces. Define what it means for a function $F : V \to W$ to be a linear transformation.
   b: Are the following linear transformations? Why? Note that the why part of the question is very important.
   b1: $F : P_2 \to P_2, p \mapsto p'' - 3p$
   b2: $F : P_2 \to \mathbb{R}, p \mapsto p(2)$
   b3: $F : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, A \to A^2 - A$.

5. Consider the differential equation $y'' + y = 0$.
   a: The functions $y_1(t) = cos(t), y_2(t) = sin(t)$ are solutions to this differential equation. Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$ that have arbitrarily many derivatives. Let $D : V \to V$ be defined by $D : f \mapsto f'' + f$. What proper;ties of D insure that the functions $a_1 y_1 + a_2 y_2$ for any $a_1, a_2 \in \mathbb{R}$ are also solutions to our differential equation?
   b: The function y(t)=t+1 is a solution to the differential equation $y'' + y = t + 1$ (We are giving you this fact; you do not have to show this is the case). What are all the solutions to the differential equation $y'' + y = t + 1$.
   c: Justify your answer to part (b) of this question.

6. a: State the rank-nullity theorem. Define dimension of a vector space. Define rank and nullity.
   b: Use the rank nullity theorem to show that any differential equation of the form

$$\frac{d^2y}{dx^2} + y = f(x)$$

   has a solution for any polynomial $f(x) \in P_2$ Here P_2 is the vector space of polynomials of degree ≤ 2.

7. Let T be the liner transformation from P_2 to P_2 given by

$$f(x) \mapsto f'' - 2f.$$

   Find the matrix of T with respect to the basis $\{1, x - 1, (x - 1)^2\}$.

8. Define an inner product (or equivalently, a dot product) on P_2 by $< f, g >= \int_0^1 fg dx$. Find an othonormal basis of P_2 with respect to this inner product.

9. The two vectors $u_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ are orthogonal.
   a: Verify this fact.
   b: Find the vector in the space spanned by u_1 and u_2 that is closest to the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

10. Using the method of expansion by minors compute the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

   Find the determinant of the matrix using row and column operations.

11. a: Define eigenvalue and eigenvector of a matrix.
b: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 7/5 & -1/5 \\ -6/5 & 8/5 \end{pmatrix}.$$

c: Find a matrix B so that $C = BAB^{-1}$ is diagonal. What is the matrix C?
Partial Answer: the eigenvalues of the matrix A are $1, 2$, The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

12. Let $C(n)$ denote the coyote population of an imaginary ecosystem after n time intervals have passed. Similarly let $R(n)$ denote the roadrunner population of this ecosystem after n time intervals. Let

$$A = \begin{pmatrix} 10/21 & 16/21 \\ -8/21 & 46/21 \end{pmatrix}.$$

Assume that

$$\begin{pmatrix} C(n+1) \\ R(n+1) \end{pmatrix} = A \begin{pmatrix} C(n) \\ R(n) \end{pmatrix}.$$

Describe how this ecosystem behaves for different positive initial values $\begin{pmatrix} R(0) \\ C(0) \end{pmatrix}$. In particular indicate for what initial values both species survive and for what initial values only one species survives in the long run.
Partial answer: The eigenvalues of the matrix A are $2/3, 2$. The corresponding eigenvectors are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1, \\ 2 \end{pmatrix}$.