ON THE MINIMAX SPHERE EVERSION

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In the late 1950's Steve Smale proved that it is possible to evert — turn inside-out — a sphere in three space. For several years it remained a mystery how to explicitly carry this out until Arnold Shapiro, and later Bernard Morin and Bill Thurston, each invented their own sphere eversions in the 1960's and '70's. This note is a little "mystory" — which rhymes with "history" — concerning a relatively new, and in some sense optimal, sphere eversion. I hope it will explain the context in which I thought up this eversion, and how the animation presented here has come to pass.

Kids climbing fences, along with engineers building mountain roads and scientists rocketing to the moon, know that the easiest way to get from one side to the other (and back again) is to follow the path which goes over the lowest place. This is so obvious to kids that they don't have a name for — or at least they don't tell their parents about — this place on a fence, but of course this place is usually called a *pass* or *saddle* on roads through the mountains.

It is precisely such a lowest energy saddle that we encounter halfway through turning a sphere inside-out via the *minimax sphere eversion*. Indeed, the minimax sphere eversion might be viewed as the "easiest" path of immersed spheres leading from a round sphere with outside-out to one with outside-in. The energy which climbing kids and road engineers care about is the height they need to go above the surrounding territory. For us mathematicians interested in everting spheres, another energy is needed: the *elastic bending energy*, which assigns to any immersed surface the integral of the square of the mean curvature. For historical reasons this energy is often called W, after Tom Willmore, who rekindled interest in W among mathematicians in the mid-1960's.

I began working with the elastic bending energy W in the early 1980's when I was a graduate student at Berkeley learning low dimensional topology from Rob Kirby, and starting to work on minimal surfaces and variational problems with Rick Schoen. I was looking for functions with nice gradient flows on the configuration spaces of embedded or immersed surfaces. This was motivated in part by Allen Hatcher's (then recent) proof of the Smale Conjecture, which in one formulation asserts that the diffeomorphism group of $\mathbb{R}^3$ is homotopy equivalent to the orthogonal group $O(3)$. An equivalent form of the Smale Conjecture is:

- The space of embedded spheres in $\mathbb{R}^3$ is contractible.

I was interested in giving an analytic proof of this with some kind of gradient flow for W, using a key fact which had just been proven by Robert Bryant:

- The only embedded W-critical sphere is round.

My strategy was to start with any embedded sphere and follow some (negative) W-gradient flow till the sphere stopped flowing, and thus, was round. The main problem

I ran into was that any reasonable W-gradient flow need not preserve embeddedness — this is because the flow corresponds to a fourth-order parabolic equation (second-order parabolic equations enjoy a maximum principle which maintains embeddedness) — and Bryant had found immersed W-critical spheres with self intersections. Since other, more subtle issues (such as perturbing W slightly to ensure certain compactness properties of the flow) also were needed, I set aside this approach to the Hatcher Theorem.

Nevertheless, there were other nice results of Bryant about immersed W-critical spheres which I wanted to understand variationally. This leads fairly directly to the idea of the minimax sphere eversion.

To begin, let me review a couple of nice properties of W, the simplest of which is:

- W is uniquely minimized by round spheres, with the value 4π; any other surface S has energy $W(S)$ greater than 4π.

Another property that W enjoys is an inequality (discovered by Peter Li and S.T. Yau, and sharpened by me) that can be used to control the complexity of the immersed surface:

- If there is a point of $\mathbb{R}^3$ through which k "sheets" of S pass, then $W(S)$ is at least $4k\pi$; the only way equality can occur here is if there is a complete minimal surface S' in R^3 with k planar ends — "sheets at ∞" — and a Möbius transformation which carries S' to S (and ∞, to the k-uple point on S).

Notice that the first property follows from the second one, where $k = 1$ and S' is a flat plane. The proof makes use of the fact that the quantity $W + 4k\pi$ (where k is the multiplicity of the surface at ∞) is invariant under Möbius transformations of $\mathbb{R}^3 \cup \infty$.

The interesting result Bryant had shown was that for immersed spheres, the lowest critical value for W is 16π, realized by a certain family of W-critical spheres with one quadruple point. By the above, each surface arises from Möbius inversion of some minimal surface in $\mathbb{R}^3$ with 4 planar ends. (Part of Bryant's proof can be simplified and unified using an abstract skew-form invented by Nick Schmitt, or even be replaced by a clever topological argument — see my 1994 GANG preprint with Schmitt on the Spinor Representation of Minimal Surfaces for arguments of each kind.)

Now I had also been aware, through conversations (around 1982) with John Hughes, who was then finishing up his Berkeley thesis, about this nice fact due to Tom Banchoff and Nelson Max:

- Every sphere eversion must pass through an immersed sphere with at least one quadruple point.

Thus, every sphere eversion must pass over the magical $W = 16\pi$ level, by the Li-Yau inequality above. And if some W-critical sphere at this 16π level were a saddle point, then we could simply flow to either side of the saddle (in the most negative Hessian direction) by a W-gradient flow, and the flow would have to proceed down to the W-minimizing round sphere on either side. Note that these two round spheres will have the opposite orientation. So, by climbing back up the (positive) W-gradient flow, over the saddle and back down the other side, one would get an *optimal* sphere eversion: the minimax sphere eversion!

End of story? Not quite.

Hughes shared with me a beautifully illustrated manuscript by George Francis about sphere eversions equivariant under all the cyclic rotation groups. This inspired me to find an infinite family of W-critical spheres (even order group) and real projective planes (odd order group), and their corresponding complete minimal surfaces with planar ends, replete with symmetric Weierstrass data — including a W-minimizing Boy's surface with 3-fold symmetry — that I wrote about in my 1987 *Bulletin of the A. M. S.* article Conformal Geometry and Complete Minimal Surfaces. At that time (summer of 1986) Michael Callahan, David Hoffman and I made some still pictures of these surfaces at pre-Silicon-Graphics era GANG.

Over the next few years these still images of eversion midpoints toured the world as part of the GANG-produced exhibit *Getting to the Surface*, but conceptually they were not very satisfying to me. I wanted to animate this minimax eversion; however, in the mid 1980's nobody that I knew had developed effective software for modelling this kind of gradient flow. And I had other mathematics to work on, so the idea sat on a shelf until....

In 1989 I was asked to help organize the Five Colleges Geometry Institute, and in particular, to direct the first summer (1991) of the research program on the topic of computation in geometric analysis. Ken Brakke and his `evolver` were star attractions, and under the prodding of several of us, including Lucas Hsu, Ivan Sterling, John Sullivan and myself, Brakke kindly agreed to let surfaces evolve according to motions other than area-gradient flow! Indeed, that summer Brakke worked out, and programmed into `evolver`, the formulas for discretized W-gradient flow. While learning to use `evolver`, we created elaborate datafiles and evolved them (almost forever) there at Five Colleges; and later, at GANG and at the Geometry Center.

A subset of us (Hsu, Sullivan and I) eventually wrote up some of our experiments minimizing W on surfaces of higher genus in the first volume of David Epstein's new journal *Experimental Mathematics* in 1992. To illustrate some of these evolutions for general mathematical audiences, Jim Hoffman and I made the video *Elastic Surfaces and Conformal Geometry*, first shown at Berkeley in October 1992. Then and there at MSRI took place the fateful conversation among Francis, Sullivan and me, where we at long last decided to animate the minimax sphere eversion!

The actual animation of the minimax eversion took "only" 3 more years — we were scooped by *Outside In* — and relied heavily upon Brakke's development of the `evolver hessian method`, which finds that negative eigendirection needed to push the eversion midpoint saddle surface — computed from my inverted Weierstass data — to each side, decreasing the W-energy to second order. The animation also owes its smoothness to Sullivan's expertise in scripting the `evolver` to saddle-flow the surface down to the minimum of W. And the final rendering was performed using NCSA's `illiview` under Francis' guidance, yielding the ultimate form you see here...

THE MINIMAX SPHERE EVERSION!

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