"Three-Manifolds According to Grisha Perelman"

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Witnesses to Mathematical History

On a damp Monday afternoon early in April, we are driving from Amherst to Cambridge and discussing the mathematics behind one of the biggest advances in the geometry and topology of manifolds ever. We hope to witness mathematical history.

The MIT lecture theater is packed. Late arrivals sit on the floor or stand at the back. Hundreds of people – young students, old professors, and many New England-area mathematicians from all fields of mathematics – await the news.

Grisha Perelman, a mathematician from the Steklov Mathematical Institute in St. Petersburg, Russia, wears a long beard and a dark gray suit. He paces before two large blackboards, waiting to deliver the lecture. It takes a few extra minutes for everyone to get settled. At last, Perelman is introduced by MIT's Victor Kac.

Perelman tests the microphone: he says that he is not good at speaking linearly, so he intends to sacrifice clarity for liveliness. The audience is amused. Then he lifts a jumbo white chalk to the blackboard, and writes the definition of the Ricci Flow, which was introduced in 1982 by Richard Hamilton.

Ricci Flow and Geometry

The Ricci Flow describes a kind of diffusion process which spreads the curvature associated with a Riemannian metric more evenly around a manifold M:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij},$$

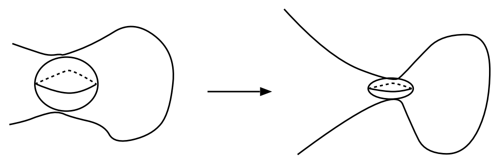
where g_{ij} denotes the Riemannian metric on M and R_{ij} denotes its Ricci curvature.

In case of a 3-manifold M, the Ricci curvature completely determines the local geometry of a metric g on M. (For example, $R_{ij} = 0$ if and only if g_{ij} is a (locally) Euclidean metric on M; this has topological consequences too: it means the universal cover of M must be Euclidean 3-space.) Under the Ricci Flow the Ricci curvature also spreads around M according to a semilinear diffusion equation.

Perelman's lecture points out a geometric interpretation of Ricci curvature: it measures the initial rate at which the area of a slice of M perpendicular to a direction changes as one parallel translates in that direction. The Ricci Flow may also be viewed as a kind of gradient flow for the metric. More details can be found in Perelman's recent preprint "The entropy formula for the Ricci flow and its geometric applications" at http://front.math.ucdavis.edu/math.DG/0303109

Ricci Flow and Topology

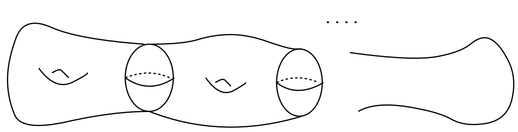
Now Perelman comes to the main application of Ricci Flow to the topology of 3-manifolds: Thurston's Geometrization Conjecture, and its corollary, the Poincare Conjecture. In 1982, Hamilton had already showed that for a 3-manifold of positive Ricci curvature, singularities in the Ricci Flow can be prevented by rescaling. Hence, the problem comes down to understanding how singularities in the Ricci Flow can develop for non-positive Ricci curvature, such as near a pinched neck:



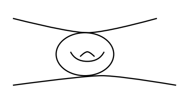
This is the type of neck found in the connected sum $M \# N$ of (connected) 3-manifolds M and N, that is, in the connected 3-manifold obtained by deleting open balls from M and N, and joining these by an $S^2 \times R$ neck.

Despite the historical importance of the moment, a few people (including one of the authors) have dozed off by the time Perelman is ready to present his main theorem:

Every closed orientable 3-manifold is a connected sum of pieces, each of which is either $S^2 \times S^1$, S^3/Γ, H^3/Γ, a graph manifold, or a collection of graph manifolds connected with finite-volume H^3/Γ's along incompressible tori.

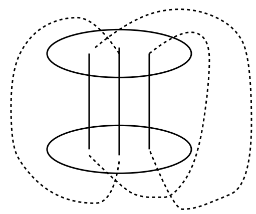


Connected sum



Splitting along an incompressible toru

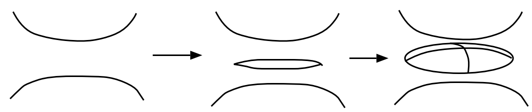
Here Γ denotes the fundamental group of the piece, and each piece of a graph manifold is Seifert-fibered:



Finally, Perelman announces that his main result implies the Poincare conjecture : a simply-connected compact 3-manifold is (topologically) S^3. The audience is now wide awake!

The Proof?

The main technique to deal with singularities is geometric surgery, which, roughly speaking, involves slitting each neck and capping-off the slit before the neck can develop into a singularity.



Key ingredients are the Bishop-Gromov theorem, new estimates on the growth rate of the Ricci curvature, and volume bound. (The details are promised in lectures 2 and 3, which we unfortunately are unable to attend: only time will tell if the proof withstands expert scrutiny.)

As the talk ends, Kac thanks the speaker – particularly, for using so little chalk! A collegial meal in Cambridge lets us digest and reflect on the events of the afternoon before our long trip back to Amherst....