THE SPINOR REPRESENTATION OF SURFACES IN SPACE

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Abstract. The spinor representation is developed for conformal immersions of Riemann surfaces into space. We adapt the approach of Dennis Sullivan [32], which treats a spin structure on a Riemann surface M as a complex line bundle S whose square is the canonical line bundle $K = T(M)$. Given a conformal immersion of M into $\mathbb{R}^3$, the unique spin strucure on S^2 pulls back via the Gauss map to a spin structure S on M, and gives rise to a pair of smooth sections (s_1, s_2) of S. Conversely, any pair of sections of S generates a (possibly periodic) conformal immersion of M under a suitable integrability condition, which for a minimal surface is simply that the spinor sections are meromorphic.

A spin structure S also determines (and is determined by) the regular homotopy class of the immersion by way of a $\mathbb{Z}_2$-quadratic form q_S. We present an analytic expression for the Arf invariant of q_S, which decides whether or not the correponding immersion can be deformed to an embedding. The Arf invariant also turns out to be an obstruction, for example, to the existence of certain complete minimal immersions.

The later parts of this paper use the spinor representation to investigate minimal surfaces with embedded planar ends. In general, we show for a spin structure S on a compact Riemann surface M with punctures at P that the space of all such (possibly periodic) minimal immersions of $M \setminus P$ into $\mathbb{R}^3$ (upto homothety) is the the product of $S^1 \times H^3$ with the Grassmanian of 2-planes in a complex vector space $\mathcal{K}$ of meromorphic sections of S. An important tool – a skew-symmetric form Ω defined by residues of a certain meromorphic quadratic differential on M – lets us compute how $\mathcal{K}$ varies as M and P are varied. Then we apply this to determine the moduli spaces of planar-ended minimal spheres and real projective planes, and also to construct a new family of minimal tori and a minimal Klein bottle with 4 ends. These surfaces compactify in S^3 to yield surfaces critical for the Möbius invariant squared mean curvature functional W. On the other hand, Robert Bryant [5] has shown all W-critical spheres and real projective planes arise this way. Thus we find at the same time the moduli spaces of W-critical spheres and real projective planes via the spinor representation.

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Contents

Introduction

In this paper we investigate the interplay between spin structures on a Riemann surface M and immersions of M into three-space. Here, a spin structure is a complex line bundle S over M such that $S \otimes S$ is the holomorphic (co)tangent bundle $T(M)$ of M. Thus we may view a section of a S as a "square root" of a $(1,0)$-form on M. Using this notion of spin structure, in the first part of this paper we develop the notion of the *spinor representation of a surface in space*, generalizing an observation of Dennis Sullivan [32]. The classical Weierstrass representation for a minimal surface is

$$(g, \eta) \longrightarrow \mathrm{Re} \int (1 - g^2, i(1 + g^2), 2g)\eta,$$

where g and η are respectively a meromorphic function and 1-form on the underlying compact Riemann surface. The spinor representation (Theorem 3) is

$$(s_1, s_2) \longrightarrow \mathrm{Re} \int (s_1^2 - s_2^2, i(s_1^2 + s_2^2), 2s_1 s_2),$$

where s_1 and s_2 are meromorphic sections of a spin structure S. Either representation gives a (weakly) conformal harmonic map $M \rightarrow \mathbb{R}^3$, which therefore parametrizes a (branched) minimal surface. In fact, either can be used to construct *any* conformal immersion (not necessarily minimal) of a surface if we relax the meromorphic condition on the data to a suitable integrability condition: in terms of spinors, we require that $(s_1,\, s_2)$ satisfy the first-order equation

$$\overline{\partial}(s_1,\, s_2) = H(|s_1|^2 + |s_2|^2)(\overline{s}_2,\, -\overline{s}_1),$$

which clearly reduces to the Cauchy-Riemann equations on a minimal surface (Theorem 4), that is, when the mean curvature H vanishes.

One feature of the spinor representation is that fundamental topological information, such as the regular homotopy class of the immersion, can be read off directly from the analytic data (Theorem 5). In fact for tori, and more generally for hyperelliptic Riemann surfaces (Theorem 21), we are able to give an explicit calculation of the Arf invariant for the immersion: the Arf invariant distinguishes whether or not an immersion of an orientable surface is regularly homotopic to an embedding. We also consider the spinor representation for nonorientable surfaces in terms of a lifting to the orientation double cover (Theorem 8). This is sufficient for constructing minimal examples later in the paper, but is less satisfying theoretically. In a future paper, we plan to consider the general case from the perspective of "pin" structures, and also give a more direct differential geometric treatment of the Arf invariant.

The second part of this paper considers finite-total-curvature minimal surfaces from the viewpoint of the spinor representation, particularly surfaces with embedded planar ends. It is well-known (see [5], [18], [19]) that such surfaces conformally compactify to give extrema for the squared mean curvature integral $W = \int H^2 dA$ popularized by Willmore. Conversely, for genus zero, all W-critical surfaces arise this way [5]. Using the spinor representation to study these special minimal surfaces has the computational advantage of converting certain quadratic conditions to linear ones. In fact, associated to a spin structure S on a closed orientable Riemann surface M is a vector space $\mathcal{K}$ of sections of S such that pairs of independent sections (s_1, s_2) from $\mathcal{K}$ form the spinor representations of all the minimal immersions of M with embedded planar ends (Theorems 9 and 10). Thus the problem of finding all these immersions is reduced to an algebraic problem (Theorem 11); to better understand $\mathcal{K}$, a skew-symmetric bilinear form Ω is defined from whose kernel $\mathcal{K}$ is computable (Definition 2 and Theorem 12).

The third (and final) part of this paper is devoted to the construction of examples and to classification results. Specifically, for a given finite topological type of surface, we explore the moduli space $\mathcal{M}$ of immersed minimal surfaces (up to similarity) of this type with embedded planar ends: the dimension and topology of $\mathcal{M}$, convergence to degenerate cases (that is, the natural closure of $\mathcal{M}$), and examples with special symmetry (which correspond to singular points of $\mathcal{M}$). The tools mentioned above permit the broad outline of a solution, but require ingenuity to apply in particular cases. For example, the form Ω allows the moduli space to be expressed as a determinantal variety which determines how the location of the ends can vary along the Riemann surface M. However, this determinantal variety is only computable when the number of ends is small. Furthermore, the basic tools, being algebraic geometrical, ignore the real analytic problems of removing periods and branch points. The latter require much subtler and often *ad hoc* methods.

Previously known results concerning minimal surfaces with embedded planar ends include the following:

- spheres exist for 4, 6, and every $n \geq 8$ ends [5], [19], [28];
- there are no immersed spheres with 3, 5, and 7 ends [6];
- the moduli spaces of immersed spheres with 4 and 6 ends, and projective planes with 3 ends have been determined [6];
- there exist rectangular tori with 4 ends [8].

Using the spinor representation we find:

- a new proof of the non-existence of spheres with 3, 5 and 7 ends is given using the skew-symmetric form Ω (Theorem 13);
- the moduli space of spheres with $2p$ ends ($2 \leq p \leq 7$) is shown to be $4(p-1)$-dimensional (Theorem 14);
- the point which compactifies the moduli space of projective planes with 3 ends is proved to be a Möbius strip, and all symmetries of these surfaces are found (Theorem 17);
- there are no three-ended tori (Theorem 18);
- there is a real two-dimensional family of four-ended immersed examples on each conformal type of torus (Theorem 19);
- there exists an immersed Klein bottle with four ends (Theorem 20).

For higher genus, the general methods we have developed here also yield (possibly branched) minimal immersions with embedded planar ends, but it becomes more and more difficult to determine precisely when branch points are absent or periods vanish: we again postpone this case to a future paper.

Most of the theorems presented here were worked out while we visited the Institute for Advanced Study during the 1992 Fall term, and were first recorded in [31]. We thank the School of Mathematics at the Institute for its hospitality, as well as A. Bobenko, G. Kamberov, P. Norman, F. Pedit, U. Pinkall, J. Richter, D. Sullivan, J. Sullivan and I. Taimanov for their comments and interest. Additionally, we should mention some more recent related developments in [4] and [15].

Part I. Spinors, Regular Homotopy Classes and the Arf Invariant

The notion of a spin structure is developed and used to describe the spinor representation of a surface in space. Section 2 defines a "quadratic form" which can be used to completely classify the spin structures on a surface, and Section 3 computes coordinates for the unique spin structure on the Riemann sphere. In the next two sections, the spinor representation of a surface is explained and related to the regular homotopy class of the surface. Section 6 shows equivalent characterizations of spin structures, the most useful of which will be that of representing spin structures by holomorphic differentials. These differentials are computed on tori (and, in Appendix B, on hyperelliptic Riemann surfaces). Section 8 takes up the question of group action on spinors, and computes the group which performs Euclidean similarity transformations. Two surfaces which are transforms of each other under the action of this group are considered to be the same. The final two sections discuss briefly the technicalities of periods and nonorientable surfaces.

1. Spin structures on a surface

A spin structure on an n-dimensional (spin) manifold M is a certain two-sheeted covering map of the $\mathrm{SO}(n)$-frame-bundle on M to a $\mathrm{Spin}(n)$-bundle (see [25], [22]). When $n = 2$, this notion of spin structure may easily be reduced to the following definition in terms of a quadratic map between complex line bundles:

$$\begin{tikzcd} S \arrow[r, "\mu"] \arrow[dr] & K = T(M) \arrow[d] \\ & M \end{tikzcd}$$

Figure 1. Spin structure

Definition 1. *A* spin structure *on a Riemann surface M is a complex line bundle S over M together with a smooth surjective fiber-preserving map $\mu : S \longrightarrow K$ to the holomorphic (co)tangent bundle $K = T(M)$ satisfying*

$$\mu(\lambda s) = \lambda^2 \mu(s) \tag{1}$$

for any section s of S. We refer to a section of S as a spinor.

Two spin structures (S, μ) and (S', μ') on a Riemann surface M are *isomorphic* if there is a line bundle isomorphism $\delta : S \longrightarrow S'$ for which $\mu = \mu'\delta$. Hence two spin structures may be isomorphic as line bundles and yet not be isomorphic as spin structures. The number of non-isomorphic spin structures on a Riemann surface M is equal to the cardinality of $H^1(M, \mathbb{Z}_2)$. (This count remains true for spin manifolds in general: see [25].) In particular, if M is a closed Riemann surface of genus g, there are $2^{2g} = \# H^1(M, \mathbb{Z}_2)$ such structures on M.

An important example is the annulus $A = \mathbb{C}^*$. There are exactly two non-isomorphic spin structures on A, which can be given explicitly as follows. The (co)tangent bundle $T(A)$ may be identified with $A \times \mathbb{C}$ by means of the global trivialization

$$a\, dz|_p \mapsto (p, a).$$

Let $S_0 = S_1 = A \times \mathbb{C}$ and define maps $\mu_k : S_k \longrightarrow T(A)$ for $k = 0, 1$ by

$$\mu_0(z, w) = (z, w^2),$$
$$\mu_1(z, w) = (z, zw^2).$$

Then (S_k, μ_k) are spin structures on A since μ_k satisfies the condition (1). Though S_0 and S_1 are isomorphic line bundles over A, they are non-isomorphic spin structures. For if S_0 and S_1 were isomorphic spin structures with bundle isomorphism $\delta : S_0 \longrightarrow S_1$ satisfying $\mu_0 = \mu_1 \delta$, then δ would be of the form $(z, w) \mapsto (z, f(z, w))$. Then $w^2 = zf^2$, implying that z has a consistent square root on $\mathbb{C}^*$, which is impossible.

2. The quadratic form associated to a spin structure

In this section, the Riemann surface M, its holomorphic (co)tangent bundle, and the spin structure are replaced with the corresponding real manifold and real vector bundles. In particular, all vector fields in this section are *real* vector fields.

To each spin structure S on the Riemann surface M we associate a $\mathbb{Z}_2$-valued quadratic form

$$q_S : H_1(M, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2.$$

To say that q_S is quadratic means that for all c_1, $c_2 \in H_1(M, \mathbb{Z}_2)$ we have

$$q_S(c_1 + c_2) = q_S(c_1) + q_S(c_2) + c_1 \cdot c_2.$$

where $c_1 \cdot c_2$ denotes the intersection number (mod 2) of c_1 with c_2.

To define $q_S(c)$, let $\alpha : S^1 \longrightarrow M$ be a smooth embedded representative of c (the existence of such an α follows from results in [24]). Let v be a smooth vector field along α which lifts to a section of S along α, and let $w(\alpha, v)$ denote the total turning number (mod 2) of the derivative vector α' against v along α. Define $q_S(c) = w(\alpha, v) + 1$.

Theorem 1. *The form $q_S : H_1(M, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ is well-defined, that is, independent of the choice of the vector field v and the choice of embedded representative α, and q_S is quadratic in the above sense.*

The proof is given in Appendix A (see also [3] and [14]).

A well-known result (see, for example, [29]) is that the equivalence class of a quadratic form $q : H_1(M, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ under linear changes of bases of $H_1(M, \mathbb{Z}_2)$ is determined by its *Arf invariant*

$$\operatorname{Arf} q = \frac{1}{\sqrt{\#H}} \sum_{\alpha \in H} (-1)^{q(\alpha)}, \tag{2}$$

where $H = H_1(M, \mathbb{Z}_2)$. The quadraticity of q insures that this invariant has values in $\{+1, -1\}$. For a compact surface of genus g, there are $2^{2g-1} + 2^{g-1}$ spin structures for which the Arf invariant of the corresponding quadratic form is $+1$, and $2^{2g-1} - 2^{g-1}$ spin structures for which it is -1 (compare Appendix B).

An alternate interpretation [2] of the Arf invariant is $\operatorname{Arf} q_S = (-1)^{\dim \mathcal{H}_S}$, where $\mathcal{H}_S$ is the space of holomorphic sections of S (also see [3]).

3. The spin structure on the Riemann sphere

The following description of the unique spin structure on S^2, as well as the spinor representation of a surface in the next section, are adapted from [32]. Identify

$$S^2 \cong [Q] = \{[z_1, z_2, z_3] \in \mathbb{CP}^2 \mid z_1^2 + z_2^2 + z_3^2 = 0\},$$

where Q is the null quadric

$$Q = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^2 + z_2^2 + z_3^2 = 0\}.$$

Then $T(S^2)$ may be identified with the restriction to $[Q]$ of the tautological line bundle

$$\mathrm{Taut}(\mathbb{CP}^2) = \{(\Lambda, x) \in \mathbb{CP}^2 \times \mathbb{C}^3 \mid x \in \Lambda\}$$

(here, $\mathbb{CP}^2$ is thought of as the lines in $\mathbb{C}^3$), so

$$T(S^2) \cong \mathrm{Taut}(\mathbb{CP}^2)|_{[Q]} = \{(\Lambda, x) \in [Q] \times Q \mid x = 0 \text{ or } \pi(x) \in \Lambda\}, \tag{3}$$

where $\pi : Q \longrightarrow [Q]$ is the canonical projection. Given this, the unique spin structure $\mathrm{Spin}(S^2)$ on S^2 may then be identified with the tautological line bundle

$$\mathrm{Spin}(S^2) \cong \mathrm{Taut}(\mathbb{CP}^1) \cong \{(\Lambda, x) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid x \in \Lambda\}, \tag{4}$$

with the associated mapping μ given by

$$\mu([z_1, z_2], (s_1, s_2)) = ([\sigma(z_1, z_2)], \sigma(s_1, s_2)),$$

where $\sigma : \mathbb{C}^2 \longrightarrow Q$ is the "Segre" map defined by

$$\sigma(z_1, z_2) = (z_1^2 - z_2^2, i(z_1^2 + z_2^2), 2z_1 z_2). \tag{5}$$

As may be checked, the map μ satisfies the conditions of Definition 1.

When $T(S^2)$ and $\mathrm{Spin}(S^2)$ are restricted respectively to their nonzero vectors and nonzero spinors, they have single coordinate charts

$$\begin{array}{rcl} \{\text{nonzero vectors in } T(S^2)\} & \longrightarrow & Q \setminus \{0\} \\ \{\text{nonzero spinors in } \mathrm{Spin}(S^2)\} & \longrightarrow & \mathbb{C}^2 \setminus \{0\} \end{array}$$

defined by taking the second component in each of (3) and (4) respectively. In this case, μ may be thought of as the two-to-one covering map $\sigma : \mathbb{C}^2 \setminus \{0\} \longrightarrow Q \setminus \{0\}$.

4. The spinor representation of a surface in space

To describe the spinor representation, let M be a connected Riemann surface with a local complex coordinate z, and $X : M \longrightarrow \mathbb{R}^3$ a conformal (but not necessarily minimal) immersion of M into space. Since X is conformal, its z-derivative $\partial X = \omega$ can be viewed as a null vector in $\mathbb{C}^3$, or via (3), as a map into the (co)tangent bundle $T(S^2)$. The Gauss map g associated to X can be viewed as a (not necessarily meromorphic) function $g : M \longrightarrow \mathbb{C} \cup \{\infty\}$ by identifying S^2 and $\mathbb{C} \cup \{\infty\}$ (via stereographic projection). This induces the bundle map (ω, g) as in the lower square of Figure 2.

The *Weierstrass representation* of the immersion X above is the pair (g, η), where g is the stereographic projection of the Gauss map, and η is the $(1,0)$-form (again, not necessarily meromorphic) on M satisfying

$$\partial X = \omega = (1 - g^2, i(1 + g^2), 2g)\, \eta.$$

Reversing this procedure (up to the problem of periods — see Section 9) one obtains the following classical result.

Theorem 2. *Given a bundle map (ω, g) of $K = T(M)$ into $T(S^2)$, if ω satisfies the integrability condition*

$$\mathrm{Re}\, d\omega = 0,$$

the $\mathbb{R}^3$-valued form $\mathrm{Re}\,\omega$ is closed (so locally exact), and thus

$$X = \mathrm{Re} \int \omega : M \longrightarrow \mathbb{R}^3$$

is a (possibly periodic, branched) conformal immersion with Gauss map g.

$$\begin{tikzcd}
S \arrow[r, "\psi"] \arrow[d, "\mu"'] & \mathrm{Spin}(S^2) \arrow[d, "\sigma"] \\
K = T(M) \arrow[r, "\omega"] \arrow[d] & T(S^2) \arrow[d] \\
M \arrow[r, "g"] & S^2
\end{tikzcd}$$

FIGURE 2. Spinor representation of a surface

The *spinor representation* of the immersion is obtained by lifting (see Figure 2) ω to the spin structures on M and S^2.

Theorem 3. *Let S be a spin structure on M, and $(\psi,\, g)$ a bundle map as in Figure 2. Assume the integrability condition. Then there exists a (possibly periodic, branched) immersion $X : M \longrightarrow \mathbb{R}^3$ with Gauss map g, whose differential $\partial X = \omega$ lifts to ψ.*

On the other hand, if $(\omega,\, g)$ is a bundle map of $K = T(M)$ into $T(S^2)$, then

(i) *there is a unique spin structure S on M such that ω lifts to a bundle map $\psi : S \longrightarrow \mathrm{Spin}(S^2)$;*

(ii) *there are exactly two such lifts ψ, and these differ only by sign.*

Proof. The integrability condition $\mathrm{Re}\, d\omega = 0$ (or its spinor equivalent, in Theorem 4 below) implies that X is well-defined up to periods.

(i). Considering $\mathrm{Spin}(S^2)$ as a $\mathbb{Z}_2$-bundle on $T(S^2)$ when restricted to nonzero spinors and vectors respectively, let S be the (unique) pullback bundle of $\mathrm{Spin}(S^2)$ under ω, and μ, ψ as shown. Extend S, ψ, and μ to include the zero spinors.

(ii). If $\iota : \mathrm{Spin}(S^2) \longrightarrow \mathrm{Spin}(S^2)$ is the order-two deck transformation for the covering $\mathrm{Spin}(S^2) \longrightarrow T(S^2)$, then $\iota \circ \psi$ is another map which in place of ψ makes the diagram commute. Conversely, if $\zeta : S \longrightarrow \mathrm{Spin}(S^2)$ is such a map, then for $x \in S$, $\zeta(x)$ is $\psi(x)$ or $\iota \circ \psi(x)$ and continuity implies that $\zeta = \psi$ or $\iota\psi$. $\square$

The Weierstrass and spinor representations are related by the equation

$$\omega = \sigma(\psi) = (s_1^2 - s_2^2, i(s_1^2 + s_2^2), 2s_1 s_2),$$

where $\psi = (s_1, s_2)$ is viewed as a pair of sections of S, and the squaring-map μ is kept implicit by writing s^2 for $\mu(s)$ and st for $\frac{1}{4}(\mu(s+t) - \mu(s-t))$. Thus these representations satisfy

$$\eta = s_1^2 \quad \text{and} \quad g = s_2/s_1.$$

How is the integrability condition expressed using spinors? We compute in the local coordinate z, where there are two sections of S whose images under μ are the (1,0)-form dz. Choose one of these sections, and refer to it consistently as $\sqrt{dz} = \varphi$. Then any spinor can be written locally in the form $s = f\varphi$, with $s^2 = \mu(s) = f^2 dz$. We define

$$\partial s = \partial f\, \varphi \quad \text{and} \quad \overline{\partial} s = \overline{\partial} f\, \varphi,$$

sections of $K \otimes S \cong S \otimes S \otimes S$ and $\overline{K} \otimes S \cong \overline{S} \otimes \overline{S} \otimes S = \overline{S} \otimes |S|^2$, respectively. For the spinor pair $\psi = (s_1, s_2)$, we also write $\not{\partial}\psi = \partial(-\overline{s}_2, \overline{s}_1)$, as suggested by Kamberov [15]; upto a conformal factor, $\not{\partial}$ is the *Dirac operator* associated to S.

The *first fundamental form* (or equivalently, the *area form*) of the conformal immersion X is a section of $|K|^2 \cong |S|^4$ given by

$$2|\omega|^2 = 4(|s_1|^2 + |s_2|^2)^2 = 4|\psi|^4.$$

The *second fundamental form* has trace-free part (the *Hopf differential*, a section of $K \otimes K$) given by

$$\Phi = \eta\, \partial g = s_1 \partial s_2 - s_2 \partial s_1 = -\psi \cdot \not{\partial}\overline{\psi}.$$

Half its trace (the *mean curvature*) is

$$H = \frac{n \cdot \overline{\partial}\omega}{|\omega|^2} = \frac{-s_1 \overline{\partial} s_2 + s_2 \overline{\partial} s_1}{(|s_1|^2 + |s_2|^2)^2} = \frac{\psi \cdot \overline{\not{\partial}\psi}}{|\psi|^4}$$

where now the Gauss map is viewed as the unit normal vector to the surface

$$n = \frac{\big(2g,\, |g|^2 - 1\big)}{|g|^2 + 1} = \frac{\big(s_1\overline{s}_2 + s_2\overline{s}_1,\, i(s_1\overline{s}_2 - s_2\overline{s}_1),\, |s_2|^2 - |s_1|^2\big)}{|s_1|^2 + |s_2|^2} \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3.$$

Differentiating the relation $\omega = \sigma(\psi)$ and using the formulas above allow us to re-express the integrability condition $\operatorname{Re} d\omega = 0$ as follows.

Theorem 4. *The integrability condition for the spinor representation $\psi = (s_1, s_2)$ is that the matrix*

$$\begin{pmatrix} s_1 & s_2 & \overline{s}_1 & \overline{s}_2 \\ -\partial\overline{s}_2 & \partial\overline{s}_1 & -\overline{\partial} s_2 & \overline{\partial} s_1 \end{pmatrix} = \begin{pmatrix} \psi & \overline{\psi} \\ \not{\partial}\psi & \overline{\not{\partial}\psi} \end{pmatrix}$$

has (real) rank 1. Equivalently, ψ satisfies a non-linear Dirac equation

$$\not{\partial}\psi = H|\psi|^2\psi$$

for some real-valued function H, necessarily the mean curvature of the surface.

Versions of this last equation have been observed also by other mathematicians, including Abresch [1], Bobenko [4], Pinkall and Richter [15], and Taimanov [17]. It is satisfied for a minimal surface ($H = 0$) if and only if $\psi = (s_1, s_2)$ is meromorphic.

5. Regular homotopy classes and spin structures

Let $X_1, X_2 : M \longrightarrow \mathbb{R}^3$ be two immersions of a surface into space. Recall the distinction between regular homotopy equivalence of the immersions X_1, X_2, and regular homotopy equivalence of the corresponding immersed surfaces — these immersed surfaces are regularly homotopic if there is a diffeomorphism h of M such that X_2 is regularly homotopic to $X_1 \circ h$ — so this latter equivalence relation is coarser.

Theorem 5. *Let $X_1, X_2 : M \longrightarrow \mathbb{R}^3$ be two immersions of a surface into space, let S_1, S_2 the spin structures induced as in Theorem* 3, *and let q_1, q_2 be the associated quadratic forms as in Theorem* 1. *Then*

(i) *X_1 and X_2 are regularly homotopic if and only if $q_1 \equiv q_2$ (mod 2).*
(ii) *The immersed surfaces $X_1(M)$ and $X_2(M)$ are regularly homotopic if and only if $\operatorname{Arf} q_1 = \operatorname{Arf} q_2$. In particular, an immersed surface is regularly homotopic to an embedding if and only if its Arf invariant equals $+1$.*

Sketch of proof. Define $\tilde{q}(\alpha)$ as half the linking number (mod 2) of the boundary curves of the image of a tubular neighborhood of α in $\mathbb{R}^3$. Then (for $i = 1, 2$)

$$q_i(\alpha) = 0 \quad \text{if and only if} \quad \begin{array}{l}\text{the Darboux frame along } \alpha \\ \text{is nontrivial as an element of} \\ \pi_1(\mathrm{SO}(3))\end{array} \quad \text{if and only if} \quad \tilde{q}_i(\alpha) = 0.$$

Hence $q_i \equiv \tilde{q}_i$ (mod 2). But X_1, X_2 are regularly homotopic if and only if $\tilde{q}_1 \equiv \tilde{q}_2$ (mod 2), and the corresponding immersed surfaces are regularly homotopic if and only if $\operatorname{Arf} \tilde{q}_1 = \operatorname{Arf} \tilde{q}_2$ (see [29]). $\square$

6. Spin structures and even-order differentials

Theorem 6 ties the notion of spin structure with other concepts from algebraic geometry. Recall that a *theta characteristic* on a Riemann surface is a divisor D such that $2D$ is the canonical divisor.

Theorem 6. *Given a Riemann surface M, there are natural bijections between the following sets of objects:*

(i) *the spin structures on M;*
(ii) *the complex line bundles S on M satisfying $S \otimes S \cong K$;*
(iii) *the theta characteristics on M;*
(iv) *the classes of non-identically-zero meromorphic differential forms on M whose zeros and poles have even orders, under the equivalence*

$$\eta_1 \sim \eta_2 \quad \textit{if and only if} \quad \begin{array}{l}\eta_1/\eta_2 = h^2 \textit{ for some meromorphic} \\ \textit{function } h \textit{ on } M.\end{array}$$

Proof. (i) if and only if (ii). Given a line bundle S on M satisfying $S \otimes S \cong K$, S is a spin structure with squaring map $\mu : S \longrightarrow S \otimes S$ defined by $\mu(s) = s \otimes s$. Conversely, given a spin structure S on M, the bundle map $\mu(s) \mapsto s \otimes s$ is well-defined and a vector-bundle isomorphism, so K is isomorphic to $S \otimes S$.

(ii) if and only if (iii). Via the natural correspondence between the line bundles on M with the divisor classes, this set of line bundles is bijective with the theta characteristics.

(iii) if and only if (iv). Again, there is a natural bijection between the meromorphic differentials with zeros and poles of even orders and the theta characteristics. Given such a differential η, the corresponding theta characteristic is $\frac{1}{2}(\eta)$. Moreover, two such differentials correspond to theta characteristics in the same linear equivalence class if and only if their ratio is the square of a meromorphic function on M. For $\eta_1/\eta_2 = h^2$ if and only if $\frac{1}{2}(\eta_1) - \frac{1}{2}(\eta_2) = (h)$. $\square$

The spin structures on a compact Riemann surface are also bijective with the various translates $\vartheta[{a_0 \atop b_0}]$ of the theta functions on the surface (see [26] for the definition of $\vartheta[{a_0 \atop b_0}]$).

7. Spin structures on tori

We compute the four spin structures on a Riemann torus T together with their values of q. Let $\mathbb{C}/\{2\omega_1, 2\omega_3\} = \mathrm{Jac}(T)$ be the Jacobian for T, and let $e_i = \wp(\omega_i)$ $(i = 1,\ 2,\ 3)$, where $\omega_2 = \omega_1 + \omega_3$. Then $h(u) = (\wp(u), \wp'(u))$ is a conformal diffeomorphism from the Jacobian to the Riemann surface M defined by $w^2 = 4(z-e_1)(z-e_2)(z-e_3)$. It is then elementary to show that the four differentials

$$\begin{aligned} du &= dz/w, \\ (\wp(u) - e_i)du &= (z-e_i)dz/w \end{aligned}$$

define the four distinct spin structures as in Theorem 6. With α_i the generator of $H_1(T, \mathbb{Z}_2)$ defined by $\alpha_i : [0,1] \longrightarrow \mathrm{Jac}(T)$, $\alpha_i(t) = 2t\omega_i$, the values of q and $\mathrm{Arf}\, q$ are tabulated.

TABLE 1. Values of q and $\mathrm{Arf}\, q$ for spin structures on tori

η	$q_\eta(0)$	$q_\eta(\alpha_1)$	$q_\eta(\alpha_2)$	$q_\eta(\alpha_3)$	$\mathrm{Arf}\, q_\eta$
du	0	1	1	1	-1
$(\wp(u) - e_1)du$	0	1	0	0	$+1$
$(\wp(u) - e_2)du$	0	0	1	0	$+1$
$(\wp(u) - e_3)du$	0	0	0	1	$+1$

An immersion corresponding to q for which $\mathrm{Arf}\, q = +1$ is regularly homotopic to the torus standardly embedded in $\mathbb{R}^3$. The value $\mathrm{Arf}\, q = -1$ corresponds to the twisted torus, which can be realized as the "diagonal" double covering of the standardly embedded torus as shown, but is not regularly homotopic to an embedding.

The more general case of hyperelliptic Riemann surfaces is considered in Appendix B, where the spin structures and their corresponding quadratic forms are computed explicitly.

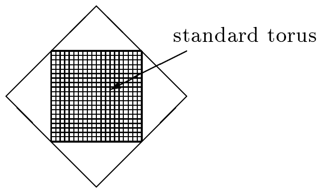

FIGURE 3. The twisted torus

8. Group action on spinors

The automorphism group of Q is the linear conformal group

$$\mathbb{C}^* \times \mathrm{SO}(3, \mathbb{C}).$$

The orbit of a conformal immersion X with $\partial X = \omega \in Q$ under this action is an 8-real-dimensional family of immersions. (This action, however, will not respect the vanishing of periods — see Section 9.) The subgroup

$$\mathbb{R}^+ \times \mathrm{SO}(3, \mathbb{R})$$

is the group of similarity transformations of Euclidean 3-space. Hence the homogeneous space

$$(\mathbb{C}^* \times \mathrm{SO}(3, \mathbb{C})) \,/ (\mathbb{R}^+ \times \mathrm{SO}(3, \mathbb{R})) \cong S^1 \times (\mathrm{SO}(3, \mathbb{C})/\mathrm{SO}(3, \mathbb{R}))\,. \tag{6}$$

is the 4-real-dimensional parameter space of non-similar surfaces in the above orbit.

In terms of spinors, we use the two-fold spin covering group $\mathrm{GL}(2, \mathbb{C})$ to get this family of immersions, as justified by the following well-known fact (see, for example, [11], [30]). Some details are given in Appendix C.

Theorem 7. *There is a unique two-fold covering homomorphism (spin)*

$$T : \mathrm{GL}(2, \mathbb{C}) \longrightarrow \mathbb{C}^* \times \mathrm{SO}(3, \mathbb{C})$$

such that for any $A \in \mathrm{GL}(2, \mathbb{C})$*,*

$$T(A)\sigma = \sigma A, \tag{7}$$

where $\sigma : \mathbb{C}^2 \longrightarrow Q$ *is as in equation (5), and* A *and* $T(A)$ *act by left multiplication on* $\mathbb{C}^2$ *and* Q *respectively. Moreover,* T *restricts to a two-fold covering of* $\mathrm{SL}(2, \mathbb{C})$, $\mathbb{R}^* \times \mathrm{SU}(2)$*, and* $\mathrm{SU}(2)$ *onto* $\mathrm{SO}(3, \mathbb{C})$, $\mathbb{R}^* \times \mathrm{SO}(3)$*, and* $\mathrm{SO}(3)$*.*

Lifting the group action on Q to an action on $\mathbb{C}^2 \setminus \{0\}$ via T, the homogeneous space in equation (6) can also be written

$$(\mathrm{GL}(2, \mathbb{C})) \,/\, (\mathbb{R}^* \times \mathrm{SU}(2)) \cong S^1 \times (\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)) \cong S^1 \times H^3, \tag{8}$$

where H^3 is hyperbolic three-space. The S^1 factor gives rise to the well-known "associate family" of minimal surfaces, which are locally isometric and share a common Gauss map. The other factor has a simple (though apparently less known) geometric interpretation as well. The Gauss map is the ratio of two spinors, so $\mathrm{SO}(3, \mathbb{C}) \cong \mathrm{PSL}(2, \mathbb{C})$ acts on the Gauss map via post-composition with a fractional linear transformation of S^2; indeed, the quotient by $\mathrm{SO}(3, \mathbb{R}) \cong \mathrm{PSU}(2)$ leaves H^3, so the second factor can be thought of as the non-rigid Möbius deformations of the Gauss map.

9. Periods

Given an immersion $X : M \longrightarrow \mathbb{R}^3$, the *period* around a simple closed curve $\gamma \subset M$ is the vector in $\mathbb{C}^3$

$$\int_\gamma \partial X.$$

If the real part of a period is not $(0,0,0)$, the resulting surface is periodic and does not have finite total curvature. It is a considerable problem to "kill the periods" — that is, choose parameters so that the integrals around every simple closed curve in M generates purely imaginary period vectors. Non-zero periods can arise along two kinds of simple closed curves:

- a simple closed curve around an end $p \in M$,
- a non-trivial simple closed curve in $H_1(M, \mathbb{Z})$.

For minimal surfaces, ∂X is meromorphic, and a simple closed curve γ around an end $p \in M$ has period

$$\int_\gamma \partial X = 2\pi i \operatorname{res}_p \partial X.$$

This integral is zero for minimal surfaces with embedded planar ends (see Section 11).

Using the spinor representation, the condition that a period along a closed curve $\gamma \subset M$ be pure imaginary can be expressed by

$$\int_\gamma \left(s_1^2 - s_2^2, i(s_1^2 + s_2^2), 2s_1 s_2\right) \in i\mathbb{R}^3,$$

equivalent to

$$\int_\gamma s_1^2 = \overline{\int_\gamma s_2^2} \quad \text{and} \quad \int_\gamma s_1 s_2 \in i\mathbb{R}. \tag{9}$$

These equations are preserved by the group $\mathbb{R}^* \times \mathrm{SU}(2)$ of homotheties.

10. Spin structures and nonorientable surfaces

To deal with immersions of a nonorientable surface M into space, we pass to the oriented two-fold cover of M. The following rather technical results are required in Part III. Without proof we state:

Lemma 1. *Let*

$A : S^2 \longrightarrow S^2$ *be the antipodal map,*
$A_* : T(S^2) \longrightarrow T(S^2)$ *the derivative of* A,
$\hat{A}_* : \mathrm{Spin}(S^2) \longrightarrow \mathrm{Spin}(S^2)$ *one of the lifts of* A_* *to Spin*(S^2) .

Then, in the coordinates of Section 4, *we have*

$$A_* = \mathrm{Conj} \qquad\qquad \hat{A}_* = \pm \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \circ \mathrm{Conj}\,.$$

$$\begin{tikzcd}
\mathrm{Spin}(S^2) \arrow[r, "\hat{A}_*"] \arrow[d] & \mathrm{Spin}(S^2) \arrow[d, "\sigma"] \\
T(S^2) \arrow[r, "A_*"] \arrow[d] & T(S^2) \arrow[d] \\
S^2 \arrow[r, "A"] & S^2
\end{tikzcd}$$

FIGURE 4. Lifts of the antipodal map

Theorem 8. *Let M be a nonorientable Riemann surface, and $X : M \longrightarrow \mathbb{R}^3$ a conformal immersion of M into space. Let $\pi : \widetilde{M} \longrightarrow M$ be an oriented double cover of M, and $\widetilde{X} = X \circ \pi$ the lift of X to this cover. Let $I : \widetilde{M} \longrightarrow \widetilde{M}$ the order-two deck transformation for the cover. With $\omega = \partial\widetilde{X}$, and in the notation of Lemma 1, we have*

(i) $gI = Ag$,
(ii) $\omega I_* = A_*\omega$,
(iii) $\psi\hat{I}_* = \pm\hat{A}_*\psi$.

We remark that the proper treatment of nonorientable surfaces should really be via "pin" structures ($\mathrm{Pin}(n)$ being the corresponding two-sheeted covering group of $\mathrm{O}(n)$), and that in this case we should have an analytic formula (in analogy with that in Appendix B for the $\mathbb{Z}_2$-valued Arf invariant on hyperelliptic surfaces) for the full $\mathbb{Z}_8$-valued Arf invariant

$$\frac{1}{\sqrt{\#H}} \sum_{\alpha \in H} i^{q(\alpha)}$$

of the associated $\mathbb{Z}_4$-valued quadratic form q on $H = H^1(M, \mathbb{Z}_2)$.

Part II. Minimal Immersions with Embedded Planar Ends

The first section of this part of our paper discusses the behavior of a minimal immersion at an embedded planar end. Lemma 2 translates this geometric behavior to a necessary and sufficient algebraic condition on the order and residue of the immersion at the end. Arising naturally from this algebraic condition is a certain vector subspace $\mathcal{K}$ of holomorphic spinors which generates all minimal surfaces with embedded planar ends (Theorem 10). More precisely, two sections chosen from $\mathcal{K}$ form the spinor representation of a minimal surface, and conversely, any such surface must arise this way (Theorem 9). However, such a surface is usually periodic, and possibly a branched immersion. In order to compute $\mathcal{K}$ explicitly, a skew-symmetric bilinear form Ω is constructed (Definition 2) whose kernel is closely related to the space $\mathcal{K}$. In Part III, this form is used to prove existence and non-existence theorems for a variety of examples.

11. Algebraic characterization of embedded planar ends

The geometric condition that an end of a minimal immersion be embedded and planar can be translated to algebraic conditions (see, for example, [7]). Let $X : D \setminus \{p\} \longrightarrow \mathbb{R}^3$ be a conformal minimal immersion of an open disk D punctured at p such that $\lim_{q\to p} |X(q)| = \infty$. The image under X of a small neighborhood of p (and by association, p itself) is what we shall refer to as an *end*. The behavior of the end is determined by the residues and the orders of the poles of ∂X at p as follows.

Let ζ_1, ζ_2, ζ_3 be defined by

$$\partial X = (\zeta_1 - \zeta_2, i(\zeta_1 + \zeta_2), 2\zeta_3).$$

The Gauss map for this immersion (see [27]) is

$$g = \zeta_3/\zeta_1 = \zeta_2/\zeta_3.$$

First note that for X to be well-defined, we must have for any closed curve γ, which winds once around p,

$$0 = \operatorname{Re} \int_\gamma \partial X = \operatorname{Re}(2\pi i \operatorname{res}_p \partial X),$$

and so $\operatorname{res}_p \partial X$ must be *real*. Assume this, and assume initially that the limiting normal to the end is upward (that is $g(p) = \infty$). In this case,

$$\operatorname{ord}_p \zeta_2 < \operatorname{ord}_p \zeta_3 < \operatorname{ord}_p \zeta_1,$$

so the first two coordinates of $X(q)$ grow faster than does the third as $q \to p$.

It follows that $\operatorname{ord}_p \zeta_2$ cannot be -1, because if it were then

$$\operatorname{res}_p \partial X = (-\operatorname{res}_p \zeta_2, i \operatorname{res}_p \zeta_2, 0)$$

would not be real. Hence $\operatorname{ord}_p \zeta_2 \leq -2$. The image under X of a small closed curve around p is a large curve which winds around the end $|\operatorname{ord}_p \zeta_2| - 1$ times. The end is embedded precisely when $\operatorname{ord}_p \zeta_2 = -2$.

If an end is embedded, its behavior is determined by the vanishing or non-vanishing of the residues of ∂X. For an embedded end, $-2 = \operatorname{ord}_p \zeta_2 < \operatorname{ord}_p \zeta_3$, so ζ_3 has either a simple pole or no pole. If ζ_3 has a simple pole (and hence also a residue), the end grows logarithmically relative to its horizontal radius and is a *catenoid* end. If ζ_3 has no pole, the end is asymptotic to a horizontal plane and is called a *planar* end. Moreover, in this latter case, $\operatorname{res}_p \zeta_2$ must vanish (again, if it did not, $\operatorname{res}_p \partial X$ would not be real), and so $\operatorname{res}_p \partial X = (0,0,0)$.

For an end in general position the same conclusions hold, because a real rotation affects neither $\operatorname{ord}_p \partial X$ nor the reality or vanishing of $\operatorname{res}_p \partial X$. In summary, we have

Lemma 2. *Let $X : D\backslash\{p\} \longrightarrow \mathbb{R}^3$ be a conformal minimal immersion of a punctured disk. Then p is an embedded planar end if and only if*

$$\operatorname{ord}_p \partial X = -2 \quad \text{and} \quad \operatorname{res}_p \partial X = 0,$$

where $\operatorname{ord}_p \partial X$ denotes the minimum order at p of the three coordinates of ∂X.

12. Embedded planar ends and spinors

The conditions in the lemma above can be translated into conditions on the spinor representation of the minimal immersion. This leads to the definition of a space $\mathcal{K}$ of spinors, pairs of which form the spinor representation satisfying the required conditions.

Throughout the rest of Part II, the following notation is fixed:

$$\begin{array}{l} M \text{ is a compact Riemann surface of genus } g, \\ K = T(M) \text{ is the canonical line bundle,} \\ S \text{ is a spin structure on } M, \\ P = [p_1] + \cdots + [p_n] \text{ is a divisor of } n \text{ distinct points.} \end{array} \tag{10}$$

The points $p_1, \ldots, p_n$ will eventually be the ends of a minimal immersion of M whose spinor representation will be a pair of sections of S.

Let $H^0(M, \mathcal{O}(S))$ and $H^0(M, \mathcal{M}(S))$ denote respectively the vector spaces of holomorphic and meromorphic sections of S. Define

(11)

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_{M,S,P} &&= \{s \in H^0(M, \mathcal{M}(S)) \mid (s) \geq -P\} \\ \mathcal{H} &= \mathcal{H}_{M,S} &&= H^0(M, \mathcal{O}(S)) \\ \mathcal{K} &= \mathcal{K}_{M,S,P} &&= \{s \in \mathcal{F} \mid \operatorname{ord}_p s \neq 0 \text{ and } \operatorname{res}_p s^2 = 0 \text{ for all } p \in \operatorname{supp} P\}. \end{aligned}$$

We remark that

$$s \in \mathcal{K} \quad \text{if and only if} \quad \begin{array}{l}\text{the constant term in the expansion of } s \text{ van-}\\ \text{ishes at each } p \in P.\end{array} \tag{12}$$

Thus $\mathcal{H}$ and $\mathcal{K}$ are linear subspaces of $\mathcal{F}$.

Theorem 9. *Let $X : M \longrightarrow \mathbb{R}^3$ be a minimal immersion with spinor representation (s_1, s_2). Then $p \in M$ is an embedded planar end if and only if s_1, $s_2 \in \mathcal{K}$ and at least one of s_1, s_2 has a pole at p.*

Proof. By Lemma 2, p is an embedded planar end if and only if

$$\operatorname{ord}_p \partial X = -2 \quad \text{and} \quad \operatorname{res}_p \partial X = 0.$$

The first of these equations is equivalent to the condition

$$s_1,\ s_2 \in \mathcal{F}, \text{ and at least one of } s_1,\ s_2 \text{ has a pole at } p.$$

Suppose that this condition is satisfied, and let z be a conformal coordinate near p with $z(p)=0$, $\varphi^2=dz$ and

$$s_1=\left(\frac{a_{-1}}{z}+a_0+\dots\right)\varphi \quad\text{and}\quad s_2=\left(\frac{b_{-1}}{z}+b_0+\dots\right)\varphi$$

be expansions of s_1 and s_2 respectively. Then the condition $\operatorname{res}_p \partial X=0$ is equivalent to

$$\begin{aligned}\operatorname{res}_p s_1^2 &= 2a_{-1}a_0=0\\ \operatorname{res}_p s_1s_2 &= a_{-1}b_0+a_0b_{-1}=0\\ \operatorname{res}_p s_2^2 &= 2b_{-1}b_0=0\end{aligned}$$

or, under the assumption that s_1 or s_2 has a pole at p (i.e., $a_{-1}\neq 0$ or $b_{-1}\neq 0$)

$$a_0=b_0=0.$$

This is to say that s_1, $s_2\in\mathcal{K}$. □

13. Moduli spaces of minimal surfaces with embedded planar ends

The spinor representation yields the following general result for a fixed spin structure S over a fixed Riemann surface M with fixed punctures at P.

Theorem 10. *Let P be a divisor and S a spin structure on M as in equation (10), and let $\mathcal{K}=\mathcal{K}_{M,S,P}$ as in equation (11) with $m=\dim\mathcal{K}\geq 2$. Then the space of complete minimal (branched, possibly periodic) immersions of M into $\mathbb{R}^3$ with finite total curvature and embedded planar ends at* $\operatorname{supp} P$ *is the complex $2(m-1)$-dimensional manifold* $\mathrm{Gr}_2(\mathcal{K})\times\left(S^1\times H^3\right)$. *All these immersions are regularly homotopic and in the class determined by S.*

Proof. Fix a point of the Grassmanian $\mathrm{Gr}_2(\mathcal{K})$, which represents a two-dimensional complex plane in $\mathcal{K}$. Let (t_1,t_2) be a basis for this plane. Then the family of (branched, possibly periodic) minimal immersions is given by $X=\operatorname{Re}F$, where

$$F=\int(s_1^2-s_2^2, i(s_1^2+s_2^2), 2s_1s_2)$$

and

$$\begin{pmatrix} s_1\\ s_2\end{pmatrix}=R\begin{pmatrix} t_1\\ t_2\end{pmatrix}$$

for some $R\in\mathrm{GL}(2,\mathbb{C})=\mathbb{C}^*\times\mathrm{SL}(2,\mathbb{C})$. The surfaces are identical (up to a rotation or dilation in space) when $R\in\mathbb{R}^*\times\mathrm{SU}(2)$. Thus a parameter space for this family of surfaces is $S^1\times H^3$ (see Section 8). □

We remark that $\mathrm{Gr}_2(\mathbb{C}^m)\times\left(S^1\times H^3\right)$ is actually a quaternionic manifold.

14. The vector spaces $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{K}$

When the number of ends exceeds the genus, the dimension of the space $\mathcal{F}$ of meromorphic sections of S (with at most simple poles at the ends) is computable, and any holomorphic section in $\mathcal{K}$ must vanish.

Theorem 11. *Let M, P, and S be as in equation (10), and $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{K}$ as in equation (11). Let $g = \operatorname{genus}(M)$ and $n = \deg P$. Then, under the assumption that $n \geq g$,*

(i) $\dim \mathcal{F} = n$;
(ii) $\mathcal{K} \cap \mathcal{H} = 0$.

Proof of (i). The dimension of $\mathcal{F}$ can be computed by means of the Riemann-Roch theorem (see, for example, [13]) which states

$$\dim H^0(M, L) - \dim H^0(M, K \otimes L^*) = \deg L - g + 1$$

for an arbitrary line bundle L. Let R be the line bundle corresponding to the divisor P, and let $L = S \otimes R$. Then:

- $H^0(M, L) \cong \mathcal{F}$ by the isomorphism $s \otimes r \mapsto s$, where r is a section of R with divisor P;
- $H^0(M, K \otimes L^*) = 0$, since $\deg(K \otimes L^*) = g - 1 - n$, which is negative by hypothesis;
- $\deg L - g + 1 = n$;

from which it follows that

$$\dim \mathcal{F} = \dim H^0(M, L) = n.$$

Proof of (ii). Let $s \in \mathcal{K} \cap \mathcal{H}$ be a section which is not identically zero. Since $s \in \mathcal{K}$, we have that $\operatorname{ord}_p s \neq 0$ for all $p \in \operatorname{supp} P$ — that is, at each such p, s has either a pole or a zero. But since $s \in \mathcal{H}$, s cannot have a pole at p, and hence has a zero, so $(s) \geq P$. Conversely, if $(s) \geq P$, then $s \in \mathcal{K} \cap \mathcal{H}$, so

$$\mathcal{K} \cap \mathcal{H} = \{s \in \mathcal{F} \mid (s) \geq P\}.$$

Thus for $s \in \mathcal{K} \cap \mathcal{H}$ not identically zero,

$$n \leq \deg s = g - 1.$$

Hence if $n \geq g$, then $\mathcal{K} \cap \mathcal{H} = 0$. □

15. A bilinear form Ω which annihilates $\mathcal{K}$

To understand the vector space $\mathcal{K} = \mathcal{K}_{M,S,P}$ more explicitly, a skew-symmetric bilinear form Ω is defined on $\mathcal{F}$ whose kernel contains $\mathcal{K}$. This form may then be used in many cases to compute $\mathcal{K}$, and thereby moduli spaces of minimal surfaces with embedded planar ends. Since, in effect, we are now interested in varying M and P, it is natural to consider quadratic differentials on M with poles at P. If Φ is such a differential (that is, a meromorphic section of $K \otimes K$) with expansion

$$\Phi = \sum_{k=-\infty}^{\infty} a_k z^k dz^2$$

at a point p on M in the conformal coordinate z with $z(p) = 0$, then the number a_{-2} is independent of this choice of coordinate, and is denoted in what follows by $\mathrm{qres}_p\,\Phi$. We shall use the Hopf differential $\Phi = s_1 \partial s_2 - s_2 \partial s_1$ from Section 4.

Definition 2. *With M, $P = \sum_{k=1}^{n} [p_k]$ and S as in equation (10) define $\Omega = \Omega_{M,P,S} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{C}$ by*

$$\Omega(s_1, s_2) = -\frac{1}{2} \sum_{p \in P} \mathrm{qres}_p (s_1 \partial s_2 - s_2 \partial s_1),$$

and define $\widehat{\Omega} : \mathcal{F} \longrightarrow \mathcal{F}^*$ by $(\widehat{\Omega}(s))(t) = \Omega(s,t)$.

Theorem 12. *With $\mathcal{H}$, $\mathcal{K}$ as in equation (11), Ω satisfies the following:*

(i) *Ω is a skew-symmetric bilinear form on $\mathcal{F}$;*
(ii) $\ker \widehat{\Omega} \supseteq \mathcal{K} + \mathcal{H}$*;*
(iii) *if $\mathcal{K} \cap \mathcal{H} = 0$, then* $\ker \widehat{\Omega} = \mathcal{K} \oplus \mathcal{H}$*;*
(iv) *if $n = \deg P \geq g =$* genus (M)*, then* $\ker \widehat{\Omega} = \mathcal{K} \oplus \mathcal{H}$.

Proof. Part (i) is immediate from the definition of Ω.

(ii) and (iii). We use a choice of coordinates to factor $\widehat{\Omega}$ into a pair of linear maps whose kernels are $\mathcal{H}$ and $\mathcal{K}$ respectively.

For each $k \in \{1, \ldots, n\}$, let z_k be a conformal coordinate in a neighborhood U_k of p_k with $z_k(p_k) = 0$. Let φ_k be a spinor on U_k with $\varphi_k^2 = dz_k$. With these choices, for any spinor s, let $\alpha_r^k(s)$ denote the coefficient of z_k^r in the local expansion of s/φ_k at p_k. That is, the expansion of s at p_k is

$$s = \left(\frac{\alpha_{-1}^k(s)}{z_k} + \alpha_0^k(s) + \ldots \right) \varphi_k.$$

Then

$$0 = \sum_{p \in P} \mathrm{res}_p\, s_1 s_2 = \sum_{k=1}^{n} \left(\alpha_{-1}^k(s_1)\alpha_0^k(s_2) + \alpha_0^k(s_1)\alpha_{-1}^k(s_2) \right)$$

and so

$$\Omega(s_1, s_2) = -\frac{1}{2} \sum_{k=1}^{n} (\alpha_{-1}^k(s_1)\alpha_0^k(s_2) - \alpha_0^k(s_1)\alpha_{-1}^k(s_2)) \tag{13}$$

$$= \sum_{k=1}^{n} \alpha_0^k(s_1)\alpha_{-1}^k(s_2) = -\sum_{k=1}^{n} \alpha_{-1}^k(s_1)\alpha_0^k(s_2). \tag{14}$$

Let the linear maps $A_r : \mathcal{F} \longrightarrow \mathbb{C}^n$ $(r = -1,\ 0)$ be defined by

$$A_r(s) = (\alpha_r^1(s), \ldots, \alpha_r^n(s)).$$

Then, identifying $(\mathbb{C}^n)^*$ with $\mathbb{C}^n$ in the natural way, Ω factors as

$$\widehat{\Omega} = (A_{-1})^* \circ A_0 = -(A_0)^* \circ A_{-1}.$$

Part (ii) then follows from the facts that

$$\mathcal{H} = \ker A_{-1} \quad \text{and} \quad \mathcal{K} = \ker A_0,$$

the latter by equation (12).

Finally, since for any linear maps $X \xrightarrow{A} Y \xrightarrow{B} Z$,

$$\dim(\ker B \circ A) = \dim(\ker A) + \dim(\text{ image } A \cap \ker B) \leq \dim(\ker A) + \dim(\ker B),$$

one has

$$\dim(\ker \widehat{\Omega}) \leq \dim(\ker A_{-1}) + \dim(\ker A_0) = \dim(\mathcal{H}) + \dim(\mathcal{K}).$$

It follows, under the assumption that $\mathcal{H} \cap \mathcal{K} = 0$, that $\ker \widehat{\Omega} = \mathcal{H} \oplus \mathcal{K}$. This proves part (iii).

(iv). This follows directly from part (iii) and Theorem 11(ii). $\square$

Part III. Classification and Examples

In the first half of Part III, the skew-symmetric form Ω developed in Part II is used to investigate minimal genus zero surfaces with embedded planar ends. The first two sections demonstrate the non-existence of examples with 2, 3, 5, or 7 ends, and the dimension of the moduli space of examples with 4, 6, 8, 10, 12 and 14 ends is computed. The following two sections compute explicitly the moduli spaces for the families with 4 and 6 ends, and in section 19, the moduli space of the three-ended projective planes is investigated.

The remaining sections are devoted to minimal immersions in the regular homotopy classes of tori and Klein bottles with embedded planar ends. In Sections 20 and 21, the skew-symmetric form Ω is computed for the twisted and the untwisted tori. This computation is then used to show the nonexistence and existence of various examples. In Section 22 it is shown that no such tori exist with three ends, and in Section 23, is found a real two-dimensional family of twisted immersions with four ends exists on each conformal type of torus. An amphichiral minimal Klein bottle with four embedded planar ends is constructed in Section 24.

All of these surfaces are found (or shown not to exist) by the following general method: after computing Ω on a simple basis, its pfaffian, which is a function of the ends, is set to zero. The resulting condition on the placement of the ends — that is, the determinantal variety — together with further conditions arising from the demand that the immersion have no periods and no branch points, forms a set of equations whose simultaneous solution (or impossibility of solution) gives the desired result.

16. Existence and non-existence of genus-zero surfaces

The non-existence of genus zero minimal unbranched immersions with 3, 5 or 7 embedded planar ends was first proved in a case-by-case manner in [6]. The following is a new proof, using the ideas of Part II.

Theorem 13. *There are no complete minimal branched or unbranched immersions of a punctured sphere into space with finite total curvature and* 2, 3, 5, *or* 7 *embedded planar ends. There exist unbranched examples with* 4, 6, *and any* $n \geq 8$ *ends.*

Proof. Examples with $2p$ ends ($p \geq 2$) are given in [19], and with $2p+1$ ends ($p \geq 4$) in [28]. For the cases $n = 3$, 5, or 7, by Lemma 3(ii) below, $2 \leq \dim \mathcal{K} \leq [\sqrt{n}] \leq 2$ (here $[q]$ denotes the greatest integer less than or equal to q), so $\dim \mathcal{K} = 2$, which contradicts Lemma 3(i) that $n - \dim \mathcal{K}$ is even. The case $n = 2$ is proved in [19] (or is proved likewise by Lemma 3 below). □

We remark that there is also a simple topological proof of the non-existence of genus zero examples with 3 ends, using ideas in [18] and [20]. The trick is to exploit the $\mathrm{SO}(3,\mathbb{C})$-action discussed in Section 8 to deform the Gauss map — on a punctured sphere with planar ends there is no period obstruction to doing this — so the compactified S^2 is in general position with a unique (tranverse) triple-point, which is impossible. (By carefully treating the periods introduced by this explicit $\mathrm{SO}(3,\mathbb{C})$ deformation of the Gauss map, the same kind of argument should generalize to exclude orientable minimal surfaces of any genus with three embedded planar ends — see Section 22 for a different proof in case of tori.)

Lemma 3. *Let P be a divisor on the Riemann sphere S^2 as in equation (10) with $n = \deg P \geq 2$, and let $\mathcal{K} = \mathcal{K}_{S^2,S,P}$ be as in equation (11). Then*

(i) $n - \dim \mathcal{K}$ *is even;*

(ii) *If there exists a complete branched or unbranched minimal immersion of S^2 into space with finite total curvature and n embedded planar ends in* $\operatorname{supp}(P)$, *then* $2 \leq \dim \mathcal{K} \leq \sqrt{n}$.

Proof of (i). By Theorem 12, $\ker \Omega = \mathcal{K} \oplus \mathcal{H}$. But $\mathcal{H} = 0$ because there are no holomorphic differentials on the sphere, so $\ker \Omega = \mathcal{K}$. Since Ω is skew-symmetric, $\operatorname{rank} \Omega = n - \dim \mathcal{K}$ is even (see Appendix D).

(ii). The sections s_1 and s_2 in the spinor representation (s_1, s_2) of such a surface are independent, showing the inequality $2 \leq \dim \mathcal{K}$. To show the other inequality, let z be the standard conformal coordinate on $S^2 = \mathbb{C} \cup \{\infty\}$, and let $P = \sum [a_i]$ (where the $a_i \in \mathbb{C}$ are distinct) be the divisor of the n ends. Let $g_1\eta, \ldots, g_m\eta$ be a basis for $\mathcal{K}$, where $\eta^2 = dz$. Define $f : S^2 = \mathbb{CP}^1 \longrightarrow \mathbb{CP}^{m-1}$ by

$$f = (g_1, \ldots, g_m).$$

Then f is well-defined and holomorphic even at the common zeros and the common poles of $g_1, \ldots, g_m$. Let

$$h(z) = \prod (z - a_i).$$

It follows from

$$(hg_i) = (h) + (\eta) + (g_i\eta) \geq (P - n[\infty]) + [\infty] - P = -(n-1)[\infty]$$

that

$$d_0 = \deg f \leq n - 1.$$

To show that f ramifies at each $a \in \operatorname{supp} P$, let $h_i(z) = (z - a)g_i(z)$. Then h_i does not have a pole at a. Moreover, since by hypothesis there exists a minimal surface with ends at $\operatorname{supp} P$, at least one of the g_i has a pole at a, so the h_i cannot all be zero at a. Hence the appropriate condition that f ramify at a is

$$\left(h_i h_j' - h_i' h_j\right)\Big|_a = 0 \text{ for all } i,\, j.$$

This is satisfied because of the condition (12) defining $\mathcal{K}$: the expansion of g_i at a is

$$g_i = \frac{c_i}{z - a} + \mathrm{o}(z - a),$$

so the expansion of h_i at a is

$$h_i = c_i + \mathrm{o}(z - a)^2,$$

and so $h_i'(a) = 0$ for all i. Since f ramifies at each $a \in \operatorname{supp} P$, we have

$$r_0 = \text{ ramification index of } f \ \geq n.$$

Now let $f_k : \mathbb{CP}^1 \longrightarrow \mathbb{P}(\Lambda^{k+1}\mathbb{C}^m)$ defined by $f_k = f \wedge f' \wedge \cdots \wedge f^{(k)}$ in $\mathbb{C}^m$ be the k^{th} associated curve of f, and use the Plücker formulas (an extension of the Riemann-Hurwitz formula — see [12]) which on $\mathbb{CP}^1$ are

$$-d_{k-1} + 2d_k - d_{k+1} - 2 = r_k,$$

where d_k is the degree of f_k, and r_k is the ramification index of f_k. In the table below, multiplying the numbers on the left by the inequalities on the right and adding yields

$$d_0 \geq (m+n)(m-1)/m.$$

But $n-1 \geq d_0$, so it follows that $n \geq m^2$. □

TABLE 2. Values for the Plücker formulas

$$\begin{array}{c|lcccccl}
m-1 & & & 2d_0 & - & d_1 & -2= & r_0 \geq n \\
m-2 & -d_0 & + & 2d_1 & - & d_2 & -2= & r_1 \geq 0 \\
\vdots & & & & \vdots & & & \vdots \quad \vdots \\
2 & -d_{m-4} & + & 2d_{m-3} & - & d_{m-2} & -2= & r_{m-3} \geq 0 \\
1 & -d_{m-3} & + & 2d_{m-2} & & & -2= & r_{m-2} \geq 0
\end{array}$$

We may now compute the moduli spaces of genus-zero examples with an even number of punctures (ends).

Theorem 14. *For each $p \geq 2$ there exists a real $4(p-1)$-dimensional family of minimal branched immersions of spheres punctured at $2p$ points with finite total curvature and embedded planar ends. For $2 \leq p \leq 7$, the moduli space of such immersions is exactly $4(p-1)$-dimensional.*

Proof. Let $P = \sum[a_i]$ be a divisor of degree $2p$ on S^2, and S the unique spin structure on S^2. Let $\mathcal{H}$ and $\mathcal{K}$ be as in equation (14.11). Then $\operatorname{pfaffian} \Omega = 0$ (see Appendix D) if and only if $\dim \mathcal{K} \geq 2$ if and only if there exists a surface with $2p$ ends at $\operatorname{supp} P$. Counting real dimensions, the space of $2p$ ends is $4p$-dimensional; the Möbius transformations of S^2 reduce the dimension by 6, and the pfaffian condition on the ends reduce the dimension by another 2, so the space of ends which admit surfaces is $(4p-8)$-dimensional. For each admissible choice of ends, by Theorem 10 there is a real $(4 \dim \mathcal{K} - 4)$-dimensional space of surfaces. Altogether, this totals $4p + 4\dim \mathcal{K} - 12$, which is at least $4p-4$ since $\dim \mathcal{K} \geq 2$.

In the case that $2 \leq p \leq 7$, by Lemma 3(ii), $2 \leq \dim \mathcal{K} \leq [\sqrt{2p}] \leq [\sqrt{14}] = 3$, so $\dim \mathcal{K}$, being even, must be exactly 2. □

17. Ω ON THE RIEMANN SPHERE

For the examples in Sections 18–19 we need to compute the skew-symmetric form Ω from Section 15 on the Riemann sphere. Let z be the standard conformal coordinate on $S^2 = \mathbb{C} \cup \{\infty\}$, and let $\varphi^2 = dz$ represent the unique spin structure on S^2. Let $P = [a_1] + \cdots + [a_{n-1}] + [\infty]$ with the $a_i \in \mathbb{C}$ distinct. We have $\mathcal{H} = 0$ since there are no holomorphic differentials on the sphere. A basis for $\mathcal{F}$ is

$$\{t_1, \ldots, t_{n-1}, t_n\} = \left\{ \frac{\varphi}{z-a_1}, \ldots, \frac{\varphi}{z-a_{n-1}}, \varphi \right\}.$$

These sections are in $\mathcal{F}$ since

$$(t_n) = -[\infty], \quad (t_i) = -[a_i],$$

and are independent because they have distinct poles, and so are a basis for $\mathcal{F}$ since $\dim \mathcal{F} = n$. By the local calculation (13) for Ω,

$$\Omega(t_i, t_j) = \begin{cases} \dfrac{1}{a_j - a_i} & (1 \le i \le n-1;\ \ 1 \le j \le n-1;\ \ i \ne j), \\ -1 & (1 \le i \le n-1;\ \ j = n), \\ 1 & (i = n;\ \ 1 \le j \le n-1), \\ 0 & (i = j). \end{cases}$$

18. Genus zero surfaces with four or six embedded planar ends

The family of minimal genus zero surfaces with four embedded planar ends was computed first in [5]. A different computation is included here for completeness, as well as an explicit computation of the family of such surfaces with six ends.

Theorem 15. *The space Σ_4 of complete minimal immersions of spheres punctured at four points into $\mathbb{R}^3$ with finite total curvature and embedded planar ends is $S^1 \times H^3$.*

Proof. Let z be the standard conformal coordinate on $S^2 = \mathbb{C} \cup \{\infty\}$. By a Möbius transformation of the Riemann sphere S^2, the ends can be normalized so that two of the ends are 0 and ∞ and the product of the other two is 1. Naming the normalized ends

$$\{a_1 = a, a_2 = 1/a, 0, \infty\},$$

the matrix for Ω in the basis

$$\left\{ \frac{1}{z - a_1}, \frac{1}{z - a_2}, \frac{1}{z}, 1 \right\}$$

is

$$\Omega = \begin{pmatrix} 0 & \frac{1}{a_2 - a_1} & -\frac{1}{a_1} & -1 \\ \frac{1}{a_1 - a_2} & 0 & -\frac{1}{a_2} & -1 \\ \frac{1}{a_1} & \frac{1}{a_2} & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

(see Section 17). The pfaffian (see Appendix D) of Ω computes to a nonzero multiple of

$$(a^2 - \sqrt{3}a + 1)(a^2 + \sqrt{3}a + 1).$$

This pfaffian must be zero in order for $\ker \Omega = \mathcal{K}$ to be at least two-dimensional and hence to generate surfaces. Setting this pfaffian to zero yields four interchangeable solutions for a, one of which is

$$a = (\sqrt{3} + i)/2.$$

With $\varphi^2 = dz$ as usual, a basis for $\mathcal{K}$ is

$$t_1 = \left(\frac{\sqrt{3}z - 1}{z(z^2 - \sqrt{3}z + 1)} \right) \varphi \quad \text{and} \quad t_2 = \left(\frac{z(z - \sqrt{3})}{z^2 - \sqrt{3}z + 1} \right) \varphi.$$

Thus there is a single $S^1 \times H^3$ family of immersions, as in Theorem 10, and these have no periods since we are on S^2. $\square$

When there are six ends, the conformal type of the domain is no longer unique:

Theorem 16. *The space Σ_6 of complete minimal immersions of spheres punctured at six points into space with finite total curvature and embedded planar ends is $V \times (S^1 \times H^3)$, where V is a complex algebraic surface.*

Proof. On the sphere $S^2 = \mathbb{C} \cup \{\infty\}$ with standard conformal coordinate z, the ends can be normalized so that two of the ends are at 0 and ∞, and the product of the remaining four ends is 1. With this normalization, let the ends be $\{a_1, a_2, a_3, a_4, 0, \infty\}$. Set

$$\begin{aligned}
\sigma_1 &= \quad a_1 + a_2 + a_3 + a_4, \\
\sigma_2 &= -(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4), \\
\sigma_3 &= \quad a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4.
\end{aligned}$$

The pfaffian (see Appendix D) of Ω is

$$\operatorname{pfaffian}\Omega = \tau_1\tau_3 + \sigma_1\sigma_3 - 20, \tag{15}$$

where

$$\tau_1 = \sigma_1^2 + 3\sigma_2 \quad \text{and} \quad \tau_3 = \sigma_3^2 + 3\sigma_2.$$

The condition that the pfaffian be 0 defines the algebraic subvariety

$$V = \{(\sigma_1, \sigma_2, \sigma_3) \subset (\mathbb{CP}^1)^3 \mid \operatorname{pfaffian}\Omega = 0\}$$

of codimension 1. Each point $(\sigma_1, \sigma_2, \sigma_3)$ of V yields a basis

$$\begin{aligned}
t_1 &= \left(\frac{b_3z^3 + b_2z^2 + b_1z + b_0}{z(z^4 - \sigma_1z^3 - \sigma_2z^2 - \sigma_3z + 1)}\right)\varphi, \\
t_2 &= \left(\frac{z(c_3z^3 + c_2z^2 + c_1z + c_0}{z^4 - \sigma_1z^3 - \sigma_2z^2 - \sigma_3z + 1}\right)\varphi,
\end{aligned}$$

for $\mathcal{K}_{(\sigma_1,\sigma_2,\sigma_3)}$ where

$$\begin{aligned}
&b_0 = \sigma_2, && c_0 = \sigma_3\tau_1 + 5\sigma_1, \\
&b_1 = -\sigma_2\sigma_3, && c_1 = \sigma_2\tau_1 - 2\sigma_1\sigma_3 - 10, \\
&b_2 = \sigma_2\tau_3 - 2\sigma_1\sigma_3 - 10, && c_2 = -\sigma_1\sigma_2, \\
&b_3 = \sigma_1\tau_3 + 5\sigma_3, && c_3 = \sigma_2,
\end{aligned}$$

and where $\varphi^2 = dz$. The $S^1 \times H^3$ family arises as before. □

That the four- and six-ended families are immersed follows from Lemma 4 below, which in turn follows directly from the definitions of the spaces in equation (10).

Lemma 4. *On the sphere with its unique spin structure S, let $P_1 = \sum[p_i]$ as in equation (10), and $P_2 = P_1 + [a]$, ($a \notin \operatorname{supp}(P_1)$). Let $\mathcal{F}_i = \mathcal{F}_{S^1,P_i,S}$ and $\mathcal{K}_i = \mathcal{K}_{S^2,S,P_i}$ $(i = 1, 2)$ as in equation (14.11). Then $\mathcal{K}_2 \cap \mathcal{F}_1 = \{s \in \mathcal{K}_1 \mid s(a) = 0\}$.*

Now, to see why this implies the above examples are immersed, let P_1 be the divisor of ends of even degree $n < 9$, and let (s_1, s_2) be the spinor representation of a minimal branched immersion. Supposing this surface is not immersed, let a be a branch point of the surface, and set $P_2 = P_1 + [a]$. Then s_1 and s_2 are independent sections in $\mathcal{K}_1$ and $s_1(a) = 0$, $s_2(a) = 0$, so by Lemma 4 above, s_1, $s_2 \in \mathcal{K}_2$. Applying Lemma 3(ii), we have that

$$2 \leq \dim \mathcal{K}_2 \leq [\sqrt{n}] \leq 2,$$

so $\dim \mathcal{K}_2 = 2$. This contradicts the fact that $n + 1 - \dim \mathcal{K}_2$ is even (Lemma 3(i)).

19. Projective planes with three embedded planar ends

It was shown in [19] that any finite-total-curvature minimal immersion of a punctured real projective plane with embedded ends has only planar ends, and has at least three of them. Hence those which are the subject of the following theorem are the examples of minimal projective planes with the fewest number of embedded ends. One method for determining the moduli space of minimally immersed projective planes punctured at three points was given in [6]. Here we provide another description of this moduli space using the spinor representation. Note that all these surfaces compactify to give surfaces minimizing $W = \int H^2 dA$ among all immersed real projective planes [18], with minimum energy $W = 12\pi$.

Theorem 17. *Let $\Pi_3 \subset \Sigma_6$ be the moduli space of complete minimal immersions of real projective planes punctured at three points with finite total curvature and embedded planar ends modulo Euclidean similarities. Then*

(i) *Π_3 is homeomorphic to a closed disk with one point M_0 removed from the boundary;*
(ii) *the point M_0 represents the Möbius strip with total curvature -6π in the sense that if $\gamma : \mathbb{R}^+ \longrightarrow \Pi_3$ is a curve with $\lim_{t\to\infty} \gamma(t) = M_0$, then there is a one-parameter family of immersions X_t parametrizing the surfaces $\gamma(t)$ such that as $t \to \infty$, X_t converges uniformly on compact sets to a parametrization of the Möbius strip;*
(iii) *the surfaces with non-trivial symmetry groups are represented by the boundary of the disk, which represents a one-parameter family of surfaces which have a line of reflective symmetry; among these, the only surfaces with larger symmetry groups (other than M_0) are two surfaces which have, respectively, the symmetry groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, and D_3, the dihedral group of order* 6.

Proof of (i). The two-sheeted covering of the projective plane is the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, with order-two orientation-reversing deck transformation $I(z) = -1/\overline{z}$. By a motion in PSU(2) the six preimages on the sphere of three points in the projective plane can be normalized as in Section 18 to be

$$\{a_1, I(a_1), a_2, I(a_2), 0, \infty\}$$

with the product of the first four equal to 1. With this choice, following the notation of Section 18, we have

$$\sigma_2 \in \mathbb{R}; \quad \sigma_3 = -\overline{\sigma}_1; \quad \tau_3 = \overline{\tau}_1.$$

For each choice of ends satisfying equation (15), up to dilations and isometries of space there is a unique minimal immersion of the projective plane, whose spinor representation is given by $\sqrt{i}(t_1, t_2)$, with t_1, t_2 as in Section 18. For if $\sqrt{i}(\hat{t}_1, \hat{t}_2)$ is the spinor representation of another immersion with the same ends, then a motion in $\mathbb{C}^* \times \mathrm{PSL}(2,\mathbb{C})$ can make $\hat{t}_1 = t_1$, and the compatibility condition in Theorem 8 forces $\hat{t}_2 = \pm t_1$. Hence the moduli space Π_3 can be parametrized as a quotient space of

$$\Gamma = \{(\sigma_1, \sigma_2) \in \mathbb{C} \times \mathbb{R} \mid \tau_1\tau_3 + \sigma_1\sigma_3 - 20 = 0, \;\; \sigma_3 = -\overline{\sigma}_1\},$$

where σ_1, σ_2, σ_3 are the symmetric polynomials of the ends defined in Section 18. The desired moduli space is a quotient space of Γ, since permutations of the ends give rise to the same surface.

Since the parameters σ_1 and σ_2 depend on the particular normalization of the ends made in Section 18, new parameters are chosen, namely the three direction cosines (c_1, c_2, c_3) of the angles between the ends 0, a_1 and a_2, viewed as vectors in $S^2 \subset \mathbb{R}^3$. With these parameters, the determinant of Ω becomes, up to a non-zero multiple,

$$(c_1^2+3)(c_2^2+3)(c_3^2+3) - 32(c_1c_2c_3+1).$$

The surface

$$\Gamma = \left\{(c_1, c_2, c_3) \in \mathbb{R}^3 \mid (c_1^2+3)(c_2^2+3)(c_3^2+3) - 32(c_1c_2c_3+1) = 0\right\}$$

in the cube

$$C = \left\{(x, y, z) \in \mathbb{R}^3 | -1 < x, y, z < 1\right\}$$

is a tetrahedron-like object but with smoothed edges and (omitted) vertices at $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$.

The moduli space Π_3 is diffeomorphic to a quotient of Γ which arises from permutations of the ends. A choice $c = (c_1, c_2, c_3)$ determines a set of six ends on the double-covering sphere. The group of rotations of the cube is the order-24 permutation group S_4 generated by two kinds of elements:

- permuting the three numbers (c_1, c_2, c_3),
- negating any two of the three numbers (c_1, c_2, c_3).

Action under this group determines the same six ends. Hence $D = \Gamma/S_4$ is a representation of the moduli space Π_3 of minimal projective planes with three embedded planar ends. D can be shown to be topologically a closed disk with the point corresponding to the corner $(1,1,1)$ of the cube removed.

Proof of (ii). The minimal Möbius strip with total curvature -6π, found in [23], has spinor representation

$$G(w)\sqrt{dw} = \sqrt{i}(-(w+1)/w^2, w-1)\sqrt{dw}$$

Let $(\sigma_1(s), \sigma_2(s)) : \mathbb{R}^+ \longrightarrow \Gamma$ be a proper curve. It follows from the reality of σ_2 that

$$\lim_{s\to\infty} \frac{1}{\sigma_1(s)} = \lim_{s\to\infty} \frac{1}{\sigma_2(s)} = \lim_{s\to\infty} \frac{\sigma_1(s)}{\sigma_2(s)} = 0,$$

and by a permutation of the ends we can assume

$$\lim_{s\to\infty} \frac{\overline{\sigma_1(s)}}{\sigma_1(s)} = 1.$$

Further,

$$\lim_{s\to\infty}\left|\frac{\tau_1(s)}{\sigma_1(s)}\right| = 1,$$

since

$$\left|\frac{\tau_1}{\sigma_1}\right|^2 = -\frac{\tau_1\tau_3}{\sigma_1\sigma_3} = 1 - \frac{20}{|\sigma_1^2|}.$$

Now choose a function $\alpha : \mathbb{R}^+ \longrightarrow S^1 \subset \mathbb{C}$ such that

$$\lim_{s\to\infty}\left(\frac{\tau_1(s)}{\sigma_1(s)} - \alpha(s)\right) = 0,$$

and so

$$\lim_{s\to\infty}\left(\frac{\tau_3(s)}{\sigma_1(s)} - \overline{\alpha(s)}\right) = 0.$$

Let X be defined by

$$X(z)\sqrt{dz} = \frac{\sqrt{i}}{\sigma_1}(t_1, t_2),$$

where t_1, t_2 are as in Section 18. A careful reparametrization and rotation of the surface generated by $X(z)\sqrt{dz}$ converges uniformly in compact sets to the Möbius strip given above: Let $z = \alpha w$, and

$$A_\alpha = \begin{pmatrix} a^{3/2} & 0 \\ 0 & \alpha^{-3/2} \end{pmatrix}.$$

Then

$$A_\alpha X(z)\sqrt{dz} = A_\alpha \sqrt{\alpha} X(\alpha w)\sqrt{dw}$$

is the appropriate reparametrization and rotation. This amounts to showing

$$\lim_{s\to\infty} A_{\alpha(s)}\sqrt{\alpha(s)}X(\alpha(s)w) = G(w)$$

uniformly in compact sets not containing the ends, which follows by a calculation using the limits above.

Proof of (iii). To find the surfaces in Π_3 which have non-trivial symmetry groups as surfaces in space, let $G = \mathbb{Z}_2 \times \mathrm{PSU}(2) \cong \mathrm{O}(3)$ be the group of conformal and anti-conformal diffeomorphisms of $\mathbb{C}\cup\{\infty\} = S^2$ with the property that any $\xi \in G$ commutes with I. Via stereographic projection, G can be thought of as the isometry group of $S^2 \subset \mathbb{R}^3$, so $\xi \in G$ satisfies $a \cdot b = \xi a \cdot \xi b$. The group of symmetries of the minimal surface in space induces a subgroup $H \subset G$ acting on the domain S^2. Moreover, the subgroup $H \subset G$ which permutes the ends is isomorphic to the subgroup $K \subseteq S_4$ which fixes the point (c_1, c_2, c_3) representing the ends, since $\xi \in H$ preserves the inner product defining the cosines c_1, c_2, c_3.

The point of all this is that the symmetry group of a surface represented by $(c_1, c_2, c_3) \in \Pi_3$ can be determined by finding the subgroup of S_4 which fixes (c_1, c_2, c_3). Using this method, the symmetric surfaces other than the Möbius strip at $(1, 1, 1)$ are

- elements of ∂D, each with a line of reflective symmetry,
- $(\sqrt{5}/3, 0, 0) \in \partial D$ with symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$,
- $(c, c, -c) \in \partial D$ with symmetry group $S_3 = D_3$

The last (and most symmetric) of these is a surface described in [19]. □

20. Ω ON THE TWISTED TORUS

For the non-example in Section 22, and for the example in Section 23, it is necessary to compute a basis for $\mathcal{F}$ for the twisted torus (see Section 7), and the matrix for Ω in this basis. On the torus $\mathbb{C}/\{2\omega_1, 2\omega_3\}$ with the standard coordinate u, let S be the spin structure corresponding to the twisted torus, that is, represented by the holomorphic differential $\varphi_0^2 = du$. Let $P = [a_1] + \cdots + [a_n]$ and set $\omega_2 = \omega_1 + \omega_3$ throughout the remainder of Part III.

To show that $\mathcal{H} = \{c\varphi_0 \mid c \in \mathbb{C}\}$, let $t \in \mathcal{H}$. Then $0 \le (t) = (t/\varphi_0) + (\varphi_0) = (t/\varphi_0)$. Hence t/φ_0 is a holomorphic function on the torus, so it is constant. A basis for $\mathcal{F}$ is $\{t_0, t_1, \ldots, t_{n-1}\}$, where

$$\begin{aligned} t_0 &= \varphi_0, \\ t_i &= (\zeta(u-a_i) - \zeta(u) + \zeta(a_i))\,\varphi_0, \\ &= \frac{1}{2}\left(\frac{\wp'(u)+\wp'(a_i)}{\wp(u)-\wp(a_i)}\right)\varphi_0 \end{aligned}$$

(see equation (19)). These are in $\mathcal{F}$ because

$$\begin{aligned} (t_0) &= 0 \ge -P, \\ (t_i) &= [x_i] + [y_i] - [a_i] - [0] \ge -P \end{aligned}$$

where x_i and y_i are the zeros of $\wp'(u) + \wp'(a_i)$ other than $-a_i$. These Sections are independent because they have distinct poles, and hence span $\mathcal{F}$ since $\dim \mathcal{F} = n$. To compute Ω in this basis, first compute the expansions of t_i at $a_0, \ldots, a_{n-1}$ (assume $i,\, j \ne 0$):

$$\begin{aligned} t_i &= (-u^{-1} + \mathrm{o}(u))\varphi_0, \\ t_i &= ((t_i/\varphi_0)(a_j) + \mathrm{o}(u))\varphi_0 \quad (i \ne j), \\ t_i &= (u-a_i)^{-1}\varphi_0. \end{aligned}$$

Using equation (13), we have

$$\Omega(t_i, t_j) = \begin{cases} \left.\dfrac{t_i}{\varphi_0}\right|_{a_j} & (i \ne 0;\ j \ne 0;\ i \ne j), \\ 0 & (\text{otherwise}). \end{cases}$$

21. Ω ON THE UNTWISTED TORI

As above, it is also necessary to exhibit a basis for $\mathcal{F}$ on the untwisted tori (see Section 7), as well as the matrix for Ω in this basis. On the torus $\mathbb{C}/\{2\omega_1, 2\omega_3\}$ with the standard conformal coordinate u, fix $r \in \{1,2,3\}$ and choose the spin structure on the untwisted torus, represented by

$$\varphi_r^2 = \frac{du}{\wp_r(u)},$$

where $\wp_r(u) = \wp(u) - \wp(\omega_r)$. Let $P = \sum [a_i]$ with the $a_i \in T \setminus \{0, \omega_r\}$ distinct.

For this choice of spin structure, $\mathcal{H} = 0$. To show this, first note first that $(\varphi_r) = [0] - [\omega_r]$. If $t \in \mathcal{H}$, then

$$0 \le (t) = (t/\varphi_r) + (\varphi_r) = (t/\varphi_r) + [0] - [\omega_r].$$

It follows that $(t/\varphi_r) \geq [\omega_r] - [0]$. But since t/φ_r is a function, the degree of its divisor is 0. Hence $(t/\varphi_r) = [\omega_r] - [0]$. But this is impossible by Abel's theorem on the torus: for an elliptic function f, if $(f) = \sum n_i[p_i]$ (as a formal sum) then $\sum n_i p_i = 0$ (as a sum in $\mathbb{C}$).

A basis for $\mathcal{F}$ is $\{t_1, \ldots, t_n\}$, where

$$\begin{aligned} t_i(u) &= (\zeta(u-a_i) - \zeta(u) - \zeta(\omega_r - a_i) + \zeta(\omega_r))\,\varphi_r \\ &= \frac{1}{2}\left(\frac{\wp_r(u)\wp_r'(a_i) + \wp_r'(u)\wp_r(a_i)}{\wp_r(a_i)(\wp_r(u) - \wp_r(a_i))}\right)\varphi_r \end{aligned}$$

(see equation (19)). These are in $\mathcal{F}$ because $(\varphi_r) = [0] - [\omega_r]$, so $(t_i) = [a_i - \omega_r] - [a_i] \geq -P$, and are independent because their poles are distinct, so they span $\mathcal{F}$ since $\dim \mathcal{F} = n$. The expansions of t_i at $a_1, \ldots, a_n$ are

$$\begin{aligned} t_i &= ((t_i/\varphi_r)(a_j) + \mathrm{o}(u - a_j))\varphi_r \quad (i \neq j), \\ t_i &= ((u-a_i)^{-1} + \mathrm{o}(u - a_i))\varphi_r. \end{aligned}$$

Using the local expression (13) for Ω, we have

$$\Omega(t_i, t_j) = \begin{cases} \left.\dfrac{t_i}{\varphi_r}\right|_{a_j} & (i \neq j), \\ 0 & (i = j). \end{cases}$$

A particularly simple situation arises when the ends come in pairs a and $-a$. Assume $n = 2m$ and $a_{m+i} = -a_i$ $(i = 1, \ldots, m)$. In this case, a simpler basis is $\{\hat{t}_1, \ldots, \hat{t}_m, \hat{t}_{m+1}, \ldots, \hat{t}_{2m}\}$, where for $1 \leq i \leq m$,

$$\begin{aligned} \hat{t}_i &= \frac{\wp_r(a_i)}{\wp_r'(a_i)}(t_i - t_{m+i})\,\varphi_r &&= \left(\frac{\wp_r(u)}{\wp_r(u) - \wp_r(a_i)}\right)\varphi_r, \\ \hat{t}_{m+i} &= (t_i + t_{m+i})\varphi_r &&= \left(\frac{\wp_r'(u)}{\wp_r(u) - \wp_r(a_i)}\right)\varphi_r. \end{aligned}$$

In this basis, the matrix for Ω becomes

$$\left(\begin{array}{c|c} 0 & W \\ \hline -W^t & 0 \end{array}\right),$$

where W is given by

$$W_{ij} = \begin{cases} \dfrac{4}{\wp_r(a_i) - \wp_r(a_j)} & (i < j), \\ \dfrac{4}{\wp_r(a_j) - \wp_r(a_i)} & (i > j), \\ \dfrac{\wp_r(a_i)^2 - c_p c_q}{\wp_r(a_i)(\wp_r(a_i) - c_p)(\wp_r(a_i) - c_q)} & (i = j) \end{cases}$$

and $c_p = e_p - e_r$, $c_q = e_q - e_r$, $\{p, q, r\} = \{1, 2, 3\}$. Note that the entries of W are entirely free of $\wp_r'$.

A useful property of the basis above is as follows: let $L : M \longrightarrow M$ be defined as $L(u) = -u$; then for $i \leq m$ and $j \geq m+1$,

$$L^*(\hat{t}_i \hat{t}_j) = \hat{t}_i \hat{t}_j,$$

so

$$\int_{\gamma_k} \hat{t}_i\hat{t}_j = \int_{\gamma_k} L^*(\hat{t}_i\hat{t}_j) = \int_{L(\gamma_k)} \hat{t}_i\hat{t}_j = -\int_{\gamma_k} \hat{t}_i\hat{t}_j,$$

and so

$$\int_{\gamma_k} \hat{t}_i\hat{t}_j = 0 \quad (i \le m;\ j \ge m+1;\ k = 1, 3).$$

22. Non-existence of tori with three planar ends

An outline of the proof of the non-existence of three-ended tori, twisted or untwisted, is given.

Theorem 18. *There does not exist a complete minimal branched immersion of a torus into space with finite total curvature and three embedded planar ends.*

Sketch of proof. The proof is divided into two cases: for the twisted torus there exist immersions with periods, but the periods cannot be made purely imaginary; for the untwisted torus, there are not even periodic examples.

First consider the more difficult case of the twisted torus. With everything as in Section 20, let $\{0, a_1, a_2\}$ be the set of ends, and let $p_i = \wp(a_i)$, $p_i' = \wp'(a_i)$. The condition $\dim \mathcal{K} \ge 2$ puts the following condition on the placement of the ends:

$$g_2 = 4(p_1^2 + p_1p_2 + p_2^2),$$

where g_2 is the constant in the differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. To see this, first note that $\ker \Omega = \mathcal{K} \oplus \mathcal{H}$ and $\dim \mathcal{H} = 1$. Hence if $\dim \mathcal{K} = 2$ then $\Omega \equiv 0$. Assume first that $a_1 + a_2 \ne 0$. Then $p_1 - p_2 \ne 0$, and the entries of Ω indicate that $p_1' + p_2' = 0$ Hence

$$(p_1')^2 = 4p_1^3 - g_2p_1 - g_3$$

and

$$(p_2')^2 = 4p_2^3 - g_2p_2 - g_3$$

are equal, and the desired condition follows. The condition also obtains in the case that $a_1 + a_2 = 0$; this can be shown as a limiting case of the above.

Changing basis now to simplify the period equations, let

$$\begin{aligned} \hat{t}_1 &= t_1 + \varepsilon t_2, \\ \hat{t}_2 &= t_1 + \varepsilon^2 t_2, \end{aligned}$$

where $\varepsilon = (-1+\sqrt{3})/2$. With γ_1, γ_3 the closed curves parallel to ω_1, ω_3 respectively (as in Theorem 22), the integrals relevant to the periods are (for $k = 1$, 3)

$$\int_{\gamma_k} \hat{t}_1^2 = -6q_1\omega_k, \quad \int_{\gamma_k} \hat{t}_1\hat{t}_2 = -6\eta_k, \quad \int_{\gamma_k} \hat{t}_2^2 = -6q_2\omega_k,$$

where

$$\begin{aligned} q_1 &= -((\varepsilon - \varepsilon^2)p_1 + (\varepsilon - 1)p_2)/3, \\ q_2 &= -((\varepsilon^2 - \varepsilon)p_1 + (\varepsilon^2 - 1)p_2)/3, \\ q_1q_2 &= (p_1^2 + p_1p_2 + p_2^2)/3 = g_2/12. \end{aligned}$$

A choice of a pair of independent Sections from $\mathcal{K}$ can be normalized by the action of $\mathbb{R}^* \times \mathrm{SU}(2)$ to be

$$\begin{aligned} s_1 &= z_1\hat{t}_1 + \hat{t}_2, \\ s_2 &= z_2\hat{t}_1, \end{aligned}$$

with z_1, $z_2 \in \mathbb{C}$. Then the period equations (9) can be written

$$\begin{aligned} \begin{pmatrix} 2z_1 \\ z_1^2 q_1 + q_2 \end{pmatrix} - B \begin{pmatrix} 0 \\ \overline{q}_1\overline{z}_2^2 \end{pmatrix} &= 0, \\ \begin{pmatrix} z_2 \\ q_1 z_1 z_2 \end{pmatrix} + B \begin{pmatrix} \overline{z}_2 \\ \overline{q}_1\overline{z}_1\overline{z}_2^2 \end{pmatrix} &= 0, \end{aligned}$$

where

$$B = A^{-1}\overline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad A = \begin{pmatrix} \eta_1 & \omega_1 \\ \eta_3 & \omega_3 \end{pmatrix}.$$

Changing from the variables (z_1, z_2) to (w, z_2), this system is equivalent to the system

$$\begin{aligned} w^2 + b^2 q_1 q_2 - d^2 &= 0, \\ 2w + 2d - b^2 q_1\overline{q}_1\overline{z}_2^2 &= 0, \\ wz_2 + \overline{z}_2 &= 0. \end{aligned}$$

From these it follows that

$$\begin{aligned} w\overline{w} - 1 &= 0, \\ aw^2 - \overline{a} &= 0, \\ -\overline{a} - ab^2 q_1 q_2 + ad^2 &= 0. \end{aligned}$$

This last condition, depending only on the conformal type of the torus and not on w, z_1, and z_2, is a degeneracy condition for the period equations. It also follows, by an examination of the sign of $a(w - \overline{a}) \in \mathbb{R}$, that

$$|a| > 1.$$

A delicate argument, which is omitted here, using the expansions [21]

$$\begin{aligned} g_2 &= \frac{\pi^4}{12\omega_1^4}\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right), \\ \eta_1 &= \frac{\pi^2}{12\omega_1}\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right), \end{aligned}$$

where

$$\sigma_k(n) = \sum_{d|n} d^k; \quad q = e^{2i\pi\tau}; \quad \tau = \omega_3/\omega_1$$

shows that the degeneracy condition is not satisfied under the constraint $|a| > 1$ over the whole moduli space of Riemann tori. Hence no examples with three ends can be found in the case of the twisted tori.

The case of the untwisted tori is much easier. Fix $r \in \{1, 2, 3\}$ and let φ_r be as in Section 21. Let $\{a_1, a_2, a_3\}$ be the ends, translated so that they avoid $\{0, \omega_r\}$, and let $\{t_1, t_2, t_3\}$ be the basis for $\mathcal{F}$ given in the same section. The condition that $\dim \mathcal{K} = \dim \ker \Omega \leq 2$ forces Ω to be zero. This means, for example, that t_1/φ_r have zeros at a_2 and a_3. But the zeros of t_1/φ_r are ω_r and $a_1 - \omega_r$, so one of a_2, a_3 has to be ω_r, contrary to the assumption. □

23. Minimal tori with four embedded planar ends

Here the existence of families of four-ended minimal tori — none of which are regularly homotopic to embedded tori — is established. These surfaces conformally compactify to yield W-critical twisted tori with $W = 16\pi$ and isolated umbilics.

Theorem 19. *For each conformal type of torus there exists a real two-dimensional family of complete minimal immersions of the torus punctured at four points into space with finite total curvature and embedded planar ends. Each of the tori is twisted.*

Proof. To exhibit the family, it is first necessary to determine the placement of the four ends. The ends in fact must be, up to a translation, at the four half-lattice points. To show this, on the torus $\mathbb{C}/\{2\omega_1, 2\omega_3\}$, assume the four ends are $\{0, a_1, a_2, a_3\}$, where a_1, a_2, a_3 are distinct points in the torus to be determined. With $\varphi_0^2 = du$, the matrix for Ω in the basis $\{t_0, t_1, t_2, t_3\} = \{\varphi_0, f_1\varphi_0, f_2\varphi_0, f_3\varphi_0\}$ of Section 20 is

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & f_1(a_2) & f_1(a_3) \\ 0 & f_2(a_1) & 0 & f_2(a_3) \\ 0 & f_3(a_1) & f_3(a_2) & 0 \end{pmatrix}.$$

If $\ker \Omega = \mathcal{H} \oplus \mathcal{K}$ is two-dimensional, then $\dim \mathcal{K} = 1$, since $\dim \mathcal{H} = 1$, so $\mathcal{K}$ is not big enough to generate a minimal surface. Hence to produce surfaces, $\operatorname{rank} \Omega$, being even, must be zero. In this case, all the entries of the above matrix are zero; a look at t_i shows that $\wp'(a_i) + \wp'(a_j) = 0$ for all $i \neq j$. It follows that $\wp'(a_1) = \wp'(a_2) = \wp'(a_3) = 0$, so $\{a_1, a_2, a_3\} = \{\omega_1, \omega_2, \omega_3\}$.

With the ends fixed at $\{0, \omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \ker \Omega = \mathcal{H} \oplus \mathcal{K}$, so $\{t_1, t_2, t_3\}$ is a basis for $\mathcal{K}$. The simple zeros and poles of t_1, t_2, and t_3 are shown in the following figure.

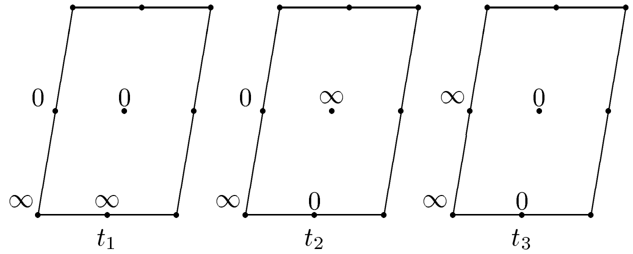

FIGURE 5. Zeros and poles of t_1, t_2, and t_3

To solve the period problem outlined in Section 9 it is convenient to choose a new basis $\{\hat{t}_1, \hat{t}_2, \hat{t}_3\}$ for $\mathcal{K}$ which "diagonalizes" the period equations. Let

$$\begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix},$$

or

$$\begin{aligned} \hat{t}_1(u) &= (\zeta(u) + \zeta(u-\omega_1) - \zeta(u-\omega_2) - \zeta(u-\omega_3) + 2\zeta(\omega_1))\varphi_0, \\ \hat{t}_2(u) &= (\zeta(u) - \zeta(u-\omega_1) + \zeta(u-\omega_2) - \zeta(u-\omega_3) + 2\zeta(\omega_2))\varphi_0, \\ \hat{t}_3(u) &= (\zeta(u) - \zeta(u-\omega_1) - \zeta(u-\omega_2) + \zeta(u-\omega_3) + 2\zeta(\omega_3))\varphi_0. \end{aligned}$$

The simple zeros and poles of $\hat{t}_1$, $\hat{t}_2$, and $\hat{t}_3$ are illustrated below.

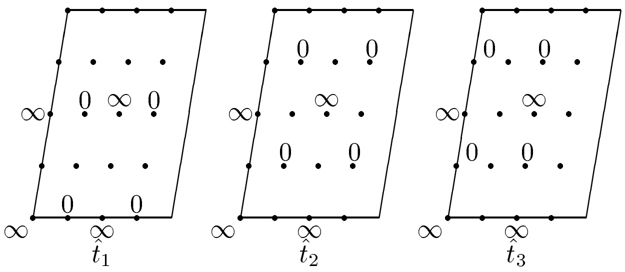

FIGURE 6. Zeros and poles of $\hat{t}_1$, $\hat{t}_2$, and $\hat{t}_3$

To compute the periods, use equation (20) to write

$$\begin{aligned} \hat{t}_i^2(u) &= (\wp(u) + \wp(u-\omega_1) + \wp(u-\omega_2) + \wp(u-\omega_3) - 4\wp(\omega_i))\,du, \\ (\hat{t}_1\hat{t}_2)(u) &= (\wp(u) - \wp(u-\omega_1) - \wp(u-\omega_2) + \wp(u-\omega_3))\,du, \\ (\hat{t}_1\hat{t}_3)(u) &= (\wp(u) - \wp(u-\omega_1) + \wp(u-\omega_2) - \wp(u-\omega_3))\,du, \\ (\hat{t}_2\hat{t}_3)(u) &= (\wp(u) + \wp(u-\omega_1) - \wp(u-\omega_2) - \wp(u-\omega_3))\,du. \end{aligned}$$

With γ_1, γ_3 the closed curves on the torus respectively parallel to ω_1, ω_3, the periods are

$$P_k^{ij} = \int_{\gamma_k} \hat{t}_i\hat{t}_j du = \begin{cases} -8(\eta_k + \omega_k e_i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (k = 1, 3),$$

where $e_i = \wp(\omega_i)$ and $\eta_k = \zeta(\omega_k)$ (see appendix E). In general, with

$$\begin{aligned} t_1 &= x_1\hat{t}_1 + x_2\hat{t}_2 + x_3\hat{t}_3 \\ t_2 &= y_1\hat{t}_1 + y_2\hat{t}_2 + y_3\hat{t}_3 \end{aligned}$$

the period equations (9) are

$$\begin{aligned} \sum_{1\le i,j\le 3} P_k^{ij} x_i x_j &= \overline{\sum_{1\le i,j\le 3} P_k^{ij} y_i y_j} \quad (k = 1, 3) \\ \sum_{1\le i,j\le 3} P_k^{ij} x_i y_j &\in i\mathbb{R} \quad (k = 1, 3). \end{aligned}$$

Now let (i,j,k) be a permutation of $(1,2,3)$ and make the particular choice

$$\begin{aligned} s_1 &= x_i\hat{t}_i + x_j\hat{t}_j, \\ s_2 &= \hat{t}_k. \end{aligned}$$

The second period equation above is satisfied for all x_i, x_j, and the first period equation can be written in the form

$$\begin{pmatrix} x_i^2 \\ x_j^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ e_i & e_j \end{pmatrix}^{-1} B \begin{pmatrix} 1 \\ \overline{e}_k \end{pmatrix}$$

where $\eta_i = \zeta(\omega_i)$ and $e_i = \wp(\omega_i)$ and B is defined in Section 22. The condition that the surface be immersed is that s_1 and s_2 have no common zeros. The zeros of s_2 are at $\{\omega_k/2, \omega_k/2+\omega_1, \omega_k/2+\omega_2, \omega_k/2+\omega_3\}$, and

$$\widehat{t}_m^2(\omega_k/2) = \widehat{t}_m^2(\omega_k/2+\omega_l) = 4(e_k - e_i) \quad (m = i,j;\ \ l = 1,2,3).$$

A necessary condition that the surface branch is that

$$(e_k - e_i)x_i^2 - (e_k - e_j)x_j^2 = 0,$$

or

$$\begin{pmatrix} g_2/2 - 3e_k^2 & -3e_k \end{pmatrix} B \begin{pmatrix} 1 \\ \overline{e}_k \end{pmatrix} = 0.$$

With the choice $\{i,j,k\} = \{1,2,3\}$ it can be shown that this condition is not satisfied in the standard fundamental region of the moduli space of tori. The proof uses the q-expansion for g_2 and η given in Section 22, as well as the expansion

$$e_1 = \frac{\pi^2}{6\omega_1^2}\left(1 + 24\sum_{n=1}^{\infty}\tau(n)q^n\right),$$

where

$$\tau(n) = \sum_{\substack{d|n \\ d \text{ odd}}} d.$$

Thus we have found a single immersion of every conformal type of torus punctured at the half-lattice points. Since the period conditions amount to at most six real conditions on 12 variables, there is a real 6-parameter family of surfaces, which modulo the action of the group in equation (8) leaves a 2-parameter family. The existence of the real two-dimensional family follows from the fact that the condition of being immersed is an open analytic condition. □

24. Minimal Klein bottles with embedded planar ends

A minimal Klein bottle is constructed in this section. Its compactification is a W-critical surface with energy $W = 16\pi$, which lies in the *amphichiral* regular homotopy class $\mathbf{K}_0 = \mathbf{B}\#\overline{\mathbf{B}}$ of Klein bottles (cf. [18], [29]). There are no minimal Klein bottles with two embedded ends [19], and we conjecture there are none with three embedded planar ends.

Theorem 20. *There exists a minimal immersion of the Klein bottle with four embedded planar ends.*

Proof. To construct this example, let $T = \mathbb{C}/\{2\omega_1, 2\omega_3\}$ be a square lattice with

$$\omega_3 = i\omega_1, \quad \omega_2 = -\omega_1 - \omega_3, \quad \wp(\omega_1) = 1, \quad \wp(\omega_2) = 0, \quad \wp(\omega_3) = -1.$$

Let $I\colon T \longrightarrow T$ be the deck transformation $I(u) = \overline{u} + \omega_1$ as in Theorem 22(i) of Appendix F. Let $a \in T$ be a point (yet to be determined) such that $I(a) = -a$, and let $E = \{a_1, \dots, a_8\} \subset T$ be the points in Table 3.

TABLE 3. Values of $\wp$ and $\wp'$ at ends of Klein bottle

u	$\wp(u)$	$\wp'(u)$	$I(u)$
$a_1 = a$	r	r'	a_5
$a_2 = a + \omega_2$	$-1/r$	r'/r^2	a_6
$a_3 = -ia$	$-r$	$-ir'$	a_4
$a_4 = -ia + \omega_2$	$1/r$	$-ir'/r^2$	a_3
$a_5 = -a_1$	r	$-r'$	a_1
$a_6 = -a_2$	$-1/r$	$-r'/r^2$	a_2
$a_7 = -a_3$	$-r$	ir'	a_8
$a_8 = -a_4$	$1/r$	ir'/r^2	a_7

We want to construct a minimal immersion $X\colon (T \setminus E)/I \longrightarrow \mathbb{R}^3$,

$$X(z) = \operatorname{Re} \int^z (s_1^2 - s_2^2, i(s_1^2 + s_2^2), 2s_1s_2),$$

where s_1, s_2 are sections of the spin structure S determined by φ, with

$$\varphi^2 = \frac{du}{\wp(u) - \wp(\omega_2)} = \frac{du}{\wp(u)}.$$

Step 1: Determination of the ends. Let $\{t_1, \dots, t_8\}$,

$$t_\alpha = \frac{\wp(u)}{\wp(u) - \wp(a_\alpha)}\varphi, \quad t_{\alpha+4} = \frac{\wp'(u)}{\wp(u) - \wp(a_\alpha)}\varphi \quad (1 \le \alpha \le 4)$$

be a basis for $\mathcal{F}$, as in Section 21. The skew-symmetric matrix for Ω in this basis is

$$\left(\begin{array}{c|c} 0 & W \\ \hline -W^t & 0 \end{array}\right),$$

where W is given by

$$W = \begin{pmatrix} \dfrac{r^2+1}{r(r^2-1)} & \dfrac{4r}{r^2+1} & \dfrac{2}{r} & \dfrac{4r}{r^2-1} \\ \dfrac{-4r}{r^2+1} & \dfrac{r(r^2+1)}{r^2-1} & \dfrac{4r}{r^2-1} & -2r \\ \dfrac{-2}{r} & \dfrac{-4r}{r^2-1} & \dfrac{-(r^2+1)}{r(r^2-1)} & \dfrac{-4r}{r^2+1} \\ \dfrac{-4r}{r^2-1} & 2r & \dfrac{4r}{r^2+1} & \dfrac{-r(r^2+1)}{r^2-1} \end{pmatrix}.$$

The desired sections s_1, s_2 lie in $\ker\Omega$, so a necessary condition for existence is that

$$0 = \det W = \frac{(3r^8 - 4r^6 + 50r^4 - 4r^2 + 3)^2}{(r^4-1)^2} = \frac{9(r^4 + mr^2 + 1)^2(r^4 + \overline{m}r^2 + 1)^2}{(r^4-1)^2},$$

where $m = -2(1-4\sqrt{2}i)/3$. Let r be the root of $r^4 + mr^2 + 1$ in the fourth quadrant; with this choice, the domain $T\backslash E$ is shown below.

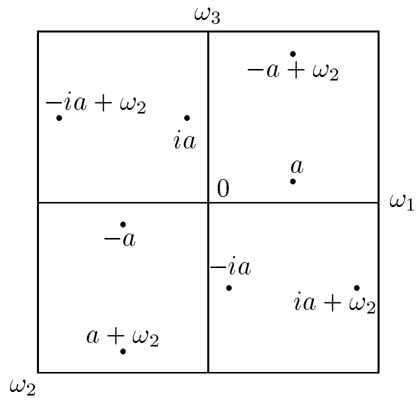

FIGURE 7. The eight ends in the double cover of the Klein bottle

Step 2: Choosing sections s_1, s_2 of S; the period equations. With r fixed as above, rank Ω is 4, and a basis for $\ker\Omega$ is $\{\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4\}$ where

$$\hat{s}_1 = \sum_{a=1}^{4} c_1^\alpha t_\alpha, \quad \hat{s}_2 = \sum_{a=1}^{4} c_2^\alpha t_\alpha, \quad \hat{s}_3 = i\overline{I^*\hat{s}_1}, \quad \hat{s}_4 = i\overline{I^*\hat{s}_2},$$

$$\begin{aligned} c_1 &= (2(r^2-1)^2, (r^2+1)(r^2-3), (r^2+1)(3r^2-1), -2(r^2-1)^2), \\ c_2 &= ((r^2+1)(3r^2-1), -2(r^2-1)^2, 2(r^2-1)^2, (r^2+1)(r^2-3)), \end{aligned}$$

and I^* is a choice of a lift of the deck transformation I to the spin structure S.

Let

$$\begin{aligned} s_1 &= x_1\hat{s}_1 + x_2\hat{s}_2 \\ s_2 &= i\overline{I^* s_1} = \overline{x}_1\hat{s}_3 + \overline{x}_2\hat{s}_4 \end{aligned} \qquad x_1, x_2 \in \mathbb{C}.$$

We want to find x_1, x_2 such that the real part of all periods are zero. By Theorem 22(iv) in Appendix F, the period equations reduce to the single equation

$$0 = \int_{\gamma_1} s_1^2 = x_1^2 P_1^{11} + 2x_1x_2P_1^{12} + x_2^2 P_1^{22},$$

where

$$P_k^{\alpha\beta} = \int_{\gamma_k} \hat{s}_\alpha \hat{s}_\beta$$

along the curve $\gamma_k : t \longmapsto t\omega_k$ $(-1 \le t \le 1)$.

Step 3: Explicit solution of the period equation. The period equation above can be solved once $P_k^{\alpha\beta}$ are known. To compute these, let

$$\hat{s}_1^2 = \frac{1}{2}\left(-\sum_{\alpha=1}^{4} A_\alpha \wp(u-a_\alpha) + B\right) du, \quad A = \sum A_\alpha$$
$$\hat{s}_1\hat{s}_2 = \frac{1}{2}\left(-\sum_{\alpha=1}^{4} C_\alpha \wp(u-a_\alpha) + D\right) du, \quad C = \sum C_\alpha$$

as in equation (20). Then

$$\begin{aligned}
P_1^{11} &= \int_{\gamma_1} \hat{s}_1^2 = A\eta_1 + B\omega_1, \\
P_1^{12} &= \int_{\gamma_1} \hat{s}_1\hat{s}_2 = C\eta_1 + D\omega_1, \\
P_3^{11} &= A\eta_3 + B\omega_3 = i(-A\eta_1 + B\omega_1) \\
P_3^{12} &= C\eta_3 + D\omega_3 = i(-C\eta_1 + D\omega_1) \\
\eta_k &= -\frac{1}{2}\int_{\gamma_k} \wp(u)du.
\end{aligned}$$

Let $J\colon T \to T$ be defined by $J(u) = iu$, and let J^* be a lift of J to S. Then

$$\hat{s}_1 = \sqrt{i}J^*\hat{s}_2, \quad \hat{s}_2 = \sqrt{i}J^*\hat{s}_1$$

for some choice of $\sqrt{i}$. Then

$$P_1^{12} = \int_{\gamma_1} \hat{s}_1\hat{s}_2 = \int_{\gamma_1} iJ^*\hat{s}_1\hat{s}_2 = i\int_{J_{(\gamma_1)}} J^*\hat{s}_1\hat{s}_2 = i\int_{\gamma_3} \hat{s}_1\hat{s}_2 = iP_3^{12},$$

so $D = 0$. Again,

$$P_1^{22} = \int_{\gamma_1} \hat{s}_2^2 = \int_{\gamma_1} iJ^*\hat{s}_1^2 = i\int_{J_{(\gamma_1)}} \hat{s}_1^2 = i\int_{\gamma_3} \hat{s}_1^2 = iP_3^{11},$$

so $P_1^{22} = A\eta_1 - B\omega_1$.

Having computed $P_1^{11}, P_1^{12}, P_1^{22}$ in terms of A, B, C, we compute A, B, C by expanding $\hat{s}_\alpha\hat{s}_\beta/du$ in two ways and equating coefficients. On the one hand, by the definition of $\hat{s}_\alpha$, we have

$$\hat{s}_a\hat{s}_\beta/du = \sum_{\gamma,\delta} \frac{c_\alpha^\gamma c_\beta^\delta \wp(u)}{(\wp(u)-\wp(a_\gamma))(\wp(u)-\wp(a_\delta))} \; (1 \le \alpha,\, \beta \le 2;\; 1 \le \gamma,\, \delta \le 4).$$

Using the formula (for $\wp'(u_0)$ finite and non-zero)

$$\frac{1}{\wp(u)-\wp(u_0)} = \frac{1/\wp'(u_0)}{u-u_0} + \cdots,$$

we get the expansion at a_γ

$$\hat{s}_a\hat{s}_\beta/du = \frac{c_a^\gamma c_\beta^\gamma \wp(a_\gamma)/(\wp'(a_\gamma))^2}{(u-a_\gamma)^2}.$$

On the other hand we have the expansions at a_γ

$$\hat{s}_1^2/du = \frac{-A_\gamma/2}{(u-a_\gamma)^2} \quad \text{and} \quad \hat{s}_1\hat{s}_2/du = \frac{-C_\gamma/2}{(u-a_\gamma)^2}.$$

Equating coefficients,

$$\begin{aligned} A_\gamma &= -2(c_1^\gamma)^2\wp(a_\gamma)/(\wp'(a_\gamma))^2 \\ C_\gamma &= -2c_1^\gamma c_2^\gamma\wp(a_\gamma)/(\wp'(a_\gamma))^2, \end{aligned}$$

so

$$\begin{aligned} A &= \sum A_i = -32r^2(r^4+4r^2+1)/3 \\ C &= \sum C_i = -2(r^4-1)^2. \end{aligned}$$

To compute B, note that s_1 has a zero at 0 to get

$$B = \sum A_\gamma\wp(a_\gamma) = 4r(r^2+1)^3.$$

This solves the period equation.

Finally, that the immersion is unbranched is the condition that s_1, s_2 have no common zeros. This amounts to the condition that if u_0 is a zero of s_1, then $I(u_0)$ is not. By using the identity

$$\overline{I^*\wp} = \frac{\wp+1}{\wp-1},$$

this can be checked by setting s_1 to zero, and solving numerically the cubic in $\wp$ which results. $\square$

Appendix A. Winding numbers and quadratic forms

In this appendix, we sketch the proof that $q_S(c) = w(\alpha, v) + 1$ defines a $\mathbb{Z}_2$-quadratic form associated to the spin structure S (Theorem 1).

Proof. Let α_0, and $\alpha_1 : S^1 \longrightarrow M$ be embedded representatives (see [24], [16]) of $c \in H_1(M, \mathbb{Z}_2)$. Let v_0, v_1 be smooth nonzero vector fields which lift along α_0, α_1 respectively to sections of the spin structure S. Let α_t ($t \in [0,1]$) be a homotopy of α_0 and α_1. Extend v_0 to a smooth nonzero vector field v in an annulus containing the image of α_t. Then $w(\alpha_t, v)$ is a continuous function of t, and an integer, hence it is constant. In particular,

$$w(\alpha_0, v_0) = w(\alpha_1, v).$$

But $v = v_0$ lifts along α_0 to a smooth section of S. So v must also lift along α_1. Since v_1 also lifts along α_1, by the lemma below

$$w(\alpha_1, v) = w(\alpha_1, v_1).$$

Thus

$$w(\alpha_0, v_0) = w(\alpha_1, v_1),$$

showing that q_S is well-defined.

Now, to show q_S is quadratic, let α_1, α_2 be embedded representatives of c_1, $c_2 \in H_1(M, \mathbb{Z}_2)$ chosen such that

$$x = \# \text{ of intersection points of } \alpha_1 \text{ and } \alpha_2 = c_1 \cdot c_2.$$

Let N be a regular neighborhood of $\alpha_1 \cup \alpha_2$ on M, with N diffeomorphic to the thrice-punctured sphere or punctured torus as $x = 0$ or 1, respectively. Choose an embedded representative $\beta : S^1 \longrightarrow N$ for $c_1 + c_2$. Therefore

$$\begin{aligned} q_S(c_1 + c_2) &= w(\beta, v) + 1 = (w(\alpha_1, v) + w(\alpha_2, v) + (x+1)) + 1 \\ &= (w(\alpha_1, v) + 1) + (w(\alpha_2, v) + 1) + x \\ &= q_S(c_1) + q_S(c_2) + c_1 \cdot c_2. \end{aligned}$$

Lemma 5. *If $\alpha : S^1 \longrightarrow M$ is an embedded curve on a surface M with spin structure S, and v_1, v_2 are smooth non-zero vector fields along α, then $w(\alpha, v_1) = w(\alpha, v_2)$ if and only if v_1 and v_2 alike lift or do not lift along α to smooth sections of S.*

Proof. We may assume M is an annulus containing $\alpha(S^1)$ as the unit circle, with spin structure S_k ($k = 0$ or 1) as in Section 1. Any vector field $S^1 \longrightarrow \mathbb{C}$ is of the form $t \mapsto t^p[f(t)]^2$, where f is smooth and

$$p = \begin{cases} k & \text{if } v \text{ lifts,} \\ 1-k & \text{if } v \text{ does not lift.} \end{cases}$$

Then, with $w_\alpha(h_1, h_2)$ defined as the winding number (mod 2) of h_1 against h_2 (or equivalently, of h_2/h_1) along α,

$$\begin{aligned} w_\alpha(v_1, v_2) &= w_\alpha(t^p[f_1(t)]^2, t^q[f_2(t)]^2) = w_\alpha(t^p, t^q) \equiv p + q \pmod 2 \\ &= \begin{cases} 0 & \text{if } v_1,\ v_2 \text{ alike lift or do not lift,} \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

But $w_\alpha(v_1, v_2) = w(\alpha, v_1) + w(\alpha, v_2)$, and the result follows. □

Appendix B. Spin structures on hyperelliptic Riemann surfaces

Here the spin structures, their corresponding quadratic forms, and their Arf invariants are computed explicitly for hyperelliptic Riemann surfaces.

Theorem 21. *Let*

$$M = \left\{ [x_1, x_2, x_3] \in \mathbb{CP}^2 \;\middle|\; x_2^2 x_3^{2g-1} = \prod_{i=1}^{2g+1}(x_1 - a_i x_3) \right\}$$

be a hyperelliptic Riemann surface of genus g, where $A = \{a_1, \ldots, a_{2g+1}\} \subset \mathbb{C}$ is a set of $2g+1$ distinct points. Let $z = x_1/x_3$ and $w = x_2/x_3$. For each subset $B \subseteq A$, define

$$f_B(z) = \prod_{a \in B}(z - a) \quad \textit{and} \quad \eta_B = f_B(z)dz/w.$$

Then

(i) *Any differential η_B represents a spin structure in the sense of Theorem* 6.
(ii) *The set of 2^{2g} meromorphic differentials*

$$\{\eta_B \mid B \subseteq A, \#B \leq g\}$$

represent the 2^{2g} distinct spin structures on M.
(iii) *With q_B the quadratic form corresponding to η_B, let γ be a curve in M whose projection to the z-plane is a Jordan curve which avoids ∞ and A, and let $C \subseteq A$ be the set of branch points which lie in the region enclosed by γ (so $\#C$ is even). Then*

$$q_B([\gamma]) = \#(B \cap C) + \frac{1}{2}\#C \ \textit{(mod 2)}.$$

(iv) *With q_B as in (iii),*

$$\operatorname{Arf} q_B = \begin{cases} +1 & \text{if} \quad 2g - 2\#B + 1 \equiv \pm 1 \pmod 8, \\ -1 & \text{if} \quad 2g - 2\#B + 1 \equiv \pm 3 \pmod 8. \end{cases}$$

Proof of (i). Let $P_i = P_{a_i} = [a_i, 0, 1]$ and $P_\infty = [0, 1, 0]$ be the branch points of the two-sheeted cover $z : M \longrightarrow \mathbb{CP}^1$. Then the divisor of η_B is

$$2\left((g - \#B - 1)P_\infty + \sum_{a \in B} P_a \right).$$

Since this divisor is even, the differential represents a spin structure by Theorem 6.

Proof of (ii). Note that there are $\binom{2g+1}{r}$ differentials in the $(r+1)^{\text{th}}$ row, totaling $\sum_{r=0}^{g}\binom{2g+1}{r} = 2^{2g}$. All but those in the last row are holomorphic.

In order to prove that these differentials represent distinct spin structures, we first compute the relations on the divisors of the form $\sum k_i P_i + k_\infty P_\infty$. Two such divisors are equivalent if and only if there is a meromorphic function M whose divisor is their difference. Since the functions w and $z - a_i$ have respective divisors

$$\begin{aligned} (w) &= P_1 + \cdots + P_{2g+1} - (2g+1)P_\infty, \\ (z - a_i) &= 2P_i - 2P_\infty, \end{aligned}$$

we have the independent relations

$$\begin{array}{rcll} P_1+\cdots+P_{2g+1} & \equiv & (2g+1)P_\infty, & \\ 2P_i & \equiv & 2P_\infty & (i=1,\ldots,2g+1). \end{array} \tag{16}$$

To show that there are no other relations independent of these, let $\sum k_iP_i+k_\infty P_\infty\equiv 0$ be a relation. Then $\sum k_i=k_\infty$, and by the relations above, we may assume each k_i is 0 or 1. Hence the general relation may be assumed to be of the form $D-dP_\infty\equiv 0$, where D is a sum of distinct $P_i\in A$, and $d=\#D$. Let h be a function with divisor $D-dP_\infty$. Since the only pole of h is at P_∞, h is a polynomial in z and w, so there are polynomial functions f_1 and f_2 of z such that

$$h(z,w)=f_1(z)+wf_2(z).$$

Then

$$2g+1\geq d=-\operatorname{ord}P_\infty h=-\operatorname{ord}P_\infty(f_1+wf_2)\geq-\operatorname{ord}P_\infty wf_2=\deg f_2+2g+1.$$

Thus $d=2g+1$, and $D=P_1+\cdots+P_{2g+1}$, so no new relation can exist.

We want to show that η_{B_1} and η_{B_2} represent identical spin structures if and only if $B_1=B_2$ or $B_1=B_2'$, where the prime notation C' designates the complement $A\setminus C$ in A. If $B_1=B_2$, then this is clear; if $B_1=B_2'$, then $\eta_{B_2}/\eta_{B_1}=(f_2/w)^2$ is a square of a meromorphic function on M, and so η_{B_1} and η_{B_2} represent the same spin structure by Theorem 6.

Conversely, suppose that η_{B_1} and η_{B_2} represent the same spin structure. Then by Theorem 6, $\eta_{B_2}/\eta_{B_1}=h^2$ for some meromorphic function h on M. But

$$2(h)=(h^2)=(\eta_{B_2}/\eta_{B_1})=2((d_2-d_1)P_\infty+D_2-D_1),$$

where $D_1=\sum_{a\in B_1}P_a$, $D_2=\sum_{a\in B_2}P_a$, $d_1=\#B_1$, and $d_2=\#B_2$. Therefore $(d_2-d_1)P_\infty+D_2-D_1\equiv 0$. By the relations (16), this divisor is equivalent to

$$\sum_{a\in B_1\circ B_2}P_a-\#(B_1\circ B_2)P_\infty,$$

where $B_1\circ B_2$ is the symmetric difference $(B_1\cup B_2)\setminus(B_1\cap B_2)$. Since the relations (16) generate all such relations, it follows that $B_1\circ B_2$ is either $\emptyset$ or A, that is that $B_1=B_2$ or $B_1=B_2'$.

Proof of (iii). It follows from the definition of $q=q_B$ that $q([\gamma])$ is the degree (mod 2) of the map $f(z)/w$ thought of as a map from the curve γ on M to $\mathbb{C}\setminus\{0\}$. Let $h=(f/w)^2$. Then

$$\deg h=\sum_{h(p)=0}\operatorname{ord}_p h+\sum_{h(p)=\infty}\operatorname{ord}_p h,$$

the sums being restricted to points within γ. This computes to

$$\deg h=\#(B\cap C)-\#(B\cup C)=2(\#(B\cap C)-\tfrac{1}{2}\#C),$$

which shows that

$$q([\gamma])=\#(B\cap C)+\tfrac{1}{2}\#C \pmod 2.$$

Proof of (iv). In order to compute $\operatorname{Arf}q$, we first compute $\sum q(\alpha)$, where α ranges over $H_1(M,\mathbb{Z}_2)$. Correspondingly, the set of branch points C in the region enclosed by α range over the subsets of A of even cardinality. Hence $\sum q(\alpha)$ is the number of such subsets for which $q(\alpha) = 1$, that is, for which

$$\#(B \cap C) - \#(B' \cap C) \equiv 2 \pmod 4.$$

The set of such subsets is

$$\{R \cup S \mid R \subseteq B, S \subseteq B', \#R - \#S \equiv 2 \pmod 4\}.$$

The cardinality of this set is

$$\sum q(\alpha) = \sum_{i-j\equiv 2} \binom{b}{i}\binom{b'}{j},$$

where $b = \#B$, $b' = \#B'$, and the sum is over i and j with $i - j \equiv 2 \pmod 4$.

To compute this sum, define

$$\xi(c,k) = \sum_{i\equiv k} \binom{c}{i}.$$

Then

$$\begin{aligned} \sum q(\alpha) &= \sum_i \binom{b}{i} \sum_{j\equiv i+2} \binom{b'}{j} = \sum_k \binom{b}{i} \xi(b', j+2) \\ &= \sum_{p=0}^{3} \sum_n \binom{b}{4n+p} \xi(b', p+2) = \sum_{p=0}^{3} \xi(b,p)\xi(b', p+2). \end{aligned}$$

Using a fact about Pascal's triangle

$$\xi(c,k) = 2^{(c-2)/2}\left(2^{(c-2)/2} + \cos\tfrac{\pi}{4}(c-2k)\right),$$

we have

$$\begin{aligned} \sum q(\alpha) &= 2^{(2g-3)/2} \sum_{p=0}^{3} \left(2^{(b-2)/2} + \cos\tfrac{\pi}{4}(b-2p)\right)\left(2^{(b'-2)/2} + \cos\tfrac{\pi}{4}(b'-2p)\right) \\ &= 2^{g-1}\left(2^g - \frac{1}{\sqrt{2}} \sum_{p=0}^{3} \cos\tfrac{\pi}{4}(b-2p)\cos\tfrac{\pi}{4}(b'-2p)\right) \\ &= 2^{g-1}\left(2^g - \sqrt{2}\cos\tfrac{\pi}{4}(2g-2b+1)\right) \\ &= \begin{cases} 2^{g-1}(2^g-1) & \text{if} \quad 2g-2b+1 \equiv \pm 1 \pmod 8, \\ 2^{g-1}(2^g+1) & \text{if} \quad 2g-2b+1 \equiv \pm 3 \pmod 8. \end{cases} \end{aligned}$$

Since $(-1)^t = 1 - 2t$ for $t = 0$ or 1,

$$\operatorname{Arf} q = \frac{1}{2^g}\sum(-1)^{q(\alpha)} = \frac{1}{2^g}(2^{2g} - 2\sum q(\alpha))$$

is $+1$ or -1 according as $2g - 2b + 1$ is ± 1 or $\pm 3 \pmod 8$. $\square$

Appendix C. Group action on spinors

In this Appendix we outline the proof that $\mathrm{GL}(2,\mathbb{C})$ is the spin covering group of the linear conformal group (Theorem 7).

Proof. Identify $\mathbb{C}^3$ with the set Γ of trace-free 2×2 complex matrices via

$$(x_1,x_2,x_3) \longleftrightarrow \begin{pmatrix} x_3 & -x_1+ix_2 \\ -x_1-ix_2 & -x_3 \end{pmatrix} = X,$$

and identify $\mathbb{R}^3 \subset \mathbb{C}^3$ with $\Gamma_{\mathbb{R}} = \{X \in \Gamma \mid X = \overline{X}^t\}$. The inner product on $\mathbb{C}^3$ becomes

$$X\cdot Y = \textstyle\sum_1^3 x_i y_i = \frac{1}{2}\operatorname{tr} XY,$$

and

$$X\cdot X = \frac{1}{2}\operatorname{tr} X^2 = -\det X,$$

so $Q \subset \mathbb{C}^3$ is identified with

$$\Gamma_Q = \{X \in \Gamma \mid \det X = 0\}.$$

Similarly, $\mathbb{C}^2$ may be identified with the set Δ of matrices of the form

$$\begin{pmatrix} x_1 & x_1 \\ x_2 & x_2 \end{pmatrix}.$$

Under these identifications the map $\sigma : \mathbb{C}^2 \longrightarrow Q$ becomes $\sigma : \Delta \longrightarrow \Gamma_Q$ given by $\sigma(X) = XJX'$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and X' denotes the classical adjoint

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

satisfying $XX' = X'X = (\det X)I$ and $(XY)' = Y'X'$.

In order to satisfy equation (7), then T must be defined, for $X \in \Gamma$, by

$$T(A)X = AXA'.$$

It follows that $T(A)$ is linear and maps Γ to itself, and that $T : \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(3,\mathbb{C})$ is a homomorphism with kernel $\{\pm I\}$. That T restricts as indicated follows from the equation

$$T(A)X \cdot T(A)Y = (\det A)^2 X\cdot Y$$

and the fact that $T(A)(\Gamma_{\mathbb{R}}) = \Gamma_{\mathbb{R}}$ for $A \in \mathbb{R}^* \times \mathrm{SU}(2)$. $\square$

Appendix D. The pfaffian

Here we recall some basic facts about skew-symmetric forms.

Definition. *A bilinear form A on a vector space V of dimension n is* skew-symmetric *if*

$$A(v_1, v_2) + A(v_2, v_1) = 0 \textit{ for all } v_1,\ v_2 \in V,$$

or alternatively, if the matrix A for A satisfies

$$A + A^t = 0.$$

The space of skew-symmetric bilinear forms is $\bigwedge^2(V^*)$. The pfaffian is a function on skew-symmetric forms whose square is the determinant.

Definition. *For $A \in \bigwedge^2(V^*)$, the* pfaffian *of A is*

$$\text{pfaffian } A = \begin{cases} \frac{1}{m!} \overbrace{(A \wedge \cdots \wedge A)}^{m \text{ times}} & \text{if } \dim(V) = 2m \text{ is even,} \\ 0 & \text{if } \dim(V) \text{ is odd.} \end{cases}$$

For a matrix (a_{ij}) of $A \in \bigwedge^2(V^*)$ in the basis $\{e_1, \ldots, e_m\}$ the pfaffians for $m = 2$, $m = 4$, and $m = 6$ are respectively

$$\begin{gathered} a_{12}, \\ a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}, \\ a_{12}a_{34}a_{56} - a_{12}a_{35}a_{46} + a_{12}a_{36}a_{45} - a_{13}a_{24}a_{56} + a_{13}a_{25}a_{46} - \\ a_{13}a_{26}a_{45} + a_{14}a_{23}a_{56} - a_{14}a_{25}a_{36} + a_{14}a_{26}a_{35} - a_{15}a_{23}a_{46} + \\ a_{15}a_{24}a_{36} - a_{15}a_{26}a_{34} + a_{16}a_{23}a_{45} - a_{16}a_{24}a_{35} + a_{16}a_{25}a_{34}. \end{gathered}$$

The general pfaffian of a $2m \times 2m$ matrix has $(2m)!/(2m!) = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)$ terms.

Lemma. *The rank of a skew-symmetric matrix is even.*

Proof. Let A be an $m \times m$ skew-symmetric matrix with rank r. The proof is by induction on m. In the case $m = 1$, then $A = (0)$ with even rank 0. Assume for some n that the lemma is true for all skew-symmetric matrices smaller than A. If n is odd, then

$$\det A = \det A^t = \det(-A) = (-1)^n \det A = -\det A,$$

so $\det A = 0$ and A has a non-zero kernel. If n is even, then A also has a non-zero kernel unless it has full — hence even — rank $r = n$. So in either case we may assume A has a non-zero kernel.

Let $v_1, \ldots, v_{n-r}$ be a basis for $\ker A$, and let $v_1, \ldots, v_{n-r},\ w_1, \ldots, w_r$ be an extension of this basis to a basis for $\mathbb{C}^n$. Let P be the $n \times n$ matrix with these vectors as columns. Then $P^t A P$ is of the form

$$P^t A P = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_0 \end{array} \right),$$

where A_0 is an $r \times r$ matrix of rank $r < n$. Moreover,

$$(P^t A P)^t = P^t A^t P = -(P^t A P),$$

so $P^t AP$, and hence A_0 is skew-symmetric. By the induction hypothesis, $r = \operatorname{rank} A$ is even, since it is the rank of the smaller skew-symmetric matrix A_0. $\square$

Appendix E. Elliptic functions

For reference, here are some standard notations and facts about elliptic functions used in this paper (see for example [9], [10]).

Lattices. A non-degenerate lattice Λ is *real* if $\Lambda = \overline{\Lambda}$. There are two kinds of real lattices:

(i) rectangular: generators $\omega_1 \in \mathbb{R}$ and $\omega_3 \in i\mathbb{R}$ can be chosen for Λ.
(ii) rhombic: generators ω_1 and $\omega_3 = \overline{\omega}_1$ can be chosen for Λ.

For any lattice with generators ω_1, ω_3, let $\omega_2 = -\omega_1 - \omega_3$.

The Weierstrass $\wp$ function: Given a lattice Λ generated by ω_1 and ω_3, the elliptic function $\wp$ on $\mathbb{C}/\Lambda$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

where

$$\begin{aligned} &e_i = \wp(\omega_i) \quad (i = 1, 2, 3), \\ &e_1 + e_2 + e_3 = 0, \\ &g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3), \\ &g_3 = 4e_1e_2e_3. \end{aligned}$$

The function $\wp$ has a double pole at 0 and two simple zeros which come together only on the square lattice; $\wp'$ has a triple pole at 0 and three simple poles at ω_1, ω_2, ω_3.

The function $\wp$ is even; $\wp'$ is odd. On a horizontal rectangular lattice, $\wp(\overline{u}) = \overline{\wp(u)}$; on a horizontal square lattice, $\wp(iu) = -\wp(u)$.

The expansion for $\wp$ at 0 is

$$\wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \dots.$$

A useful property of $\wp$ is the following special case of the addition formula ($\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$):

$$\wp(u \pm \omega_i) = e_i + \frac{(e_i - e_j)(e_i - e_k)}{\wp(u) - e_i}. \tag{17}$$

The Weierstrass ζ function: The ζ function is defined by

$$\zeta(u) = -\int \wp(u)du,$$

with the constant of integration chosen so that $\lim_{u\to 0} \zeta(u) - u^{-1} = 0$. With $\eta_i = \zeta(\omega_i)$ $(i = 1, 2, 3)$, properties of ζ include:

$$\begin{aligned} &\eta_1 + \eta_2 + \eta_3 = 0, \\ &\zeta(u + 2\omega_i) = \zeta(u) + 2\eta_i \quad (i = 1, 2, 3), \\ &\zeta \text{ is an odd function.} \end{aligned}$$

Legendre's relation is that

$$\eta_1\omega_3 - \eta_3\omega_1 = i\pi/2. \tag{18}$$

A form of the quasi-addition formula for ζ is

$$\zeta(u-v) - \zeta(u) + \zeta(v) = \frac{1}{2}\left(\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}\right). \tag{19}$$

A useful property of elliptic functions which can also be stated in more generality is the following: Let f be an elliptic function with poles of order at most 2, with no residues, and with principal parts

$$\frac{a_1}{(u-\alpha_1)^2}, \ldots, \frac{a_n}{(u-\alpha_n)^2}.$$

Then

$$f(u) = b + \sum a_i \wp(u - a_i) \tag{20}$$

for some b, because the difference $f(u) - \sum \alpha_i \wp(u-\alpha_i)$ has no poles and hence is constant.

Appendix F. Klein bottles: conformal type, spin structure, periods

Here we show that the torus covering a Klein bottle must have the conformal type of the complex plane modulo a rectangular lattice, we compute the order-two deck transformation for the covering, and we show that the spin structure on such a torus must be untwisted. (This can also be seen from purely topological considerations.)

Theorem 22. *Let $X : K' \longrightarrow \mathbb{R}^3$ be a complete minimal immersion of a punctured Klein bottle with finite total curvature, $\pi : T \longrightarrow K = \overline{K'}$ the oriented two-sheeted covering by a torus T, and $I : T \longrightarrow T$ the order-two orientation-reversing deck transformation for this cover. Then we have the following.*

(i) *T is conformally equivalent to $\mathbb{C}/\Lambda$, where Λ is a rectangular lattice with generators $2\omega_1 \in \mathbb{R}$ and $2\omega_3 \in i\mathbb{R}$.*

(ii) *On this torus, the deck transformation I may be chosen to be $I(u) = \overline{u} + \omega_1$.*

(iii) *With this choice, the admissible spin structures are those represented by $(\wp(u) - \wp(\omega_2))du$ and $(\wp(u) - \wp(\omega_3))du$.*

(iv) *If (s_1, s_2) is the spinor representation of $X \circ \pi$ on T, the period conditions reduce to the conditions $\int_{\gamma_1} s_1^2 = 0$ and $\int_{\gamma_1} s_1 s_2 = 0$ along a closed curve γ_1 parallel to ω_1.*

Proof of (i) and (ii). Let Λ_0 be a lattice such that $T = \mathbb{C}/\Lambda_0$. Since every conformal map from T to T must be linear in the standard coordinate u on $\mathbb{C}$ and since I is anti-conformal, $I(u) = \alpha\overline{u} + \beta$ for some α, $\beta \in \mathbb{C}$. The periodicity of I and I^{-1} implies that $\alpha\overline{\Lambda_0} \subseteq \Lambda_0$ and $\overline{\alpha}^{-1}\overline{\Lambda_0} \subseteq \Lambda_0$. These together imply that $\alpha\overline{\Lambda_0} = \Lambda_0$. Choose $\gamma \in \mathbb{C}$ satisfying $|\gamma| = 1$ and $\overline{\gamma}/\gamma = \alpha$; the rotated lattice $\Lambda = \gamma\Lambda_0$ satisfies $\overline{\Lambda} = \Lambda$ (a so-called *real* lattice). Hence Λ is either rectangular with generators $2\omega_1 \in \mathbb{R}$, $2\omega_3 \in i\mathbb{R}$, or Λ is rhombic with generators $2\omega_1$ and $2\omega_3 = 2\overline{\omega}_1$. On $\mathbb{C}/\Lambda$

we have $I(u) = \alpha\overline{u} + \beta$ for some new α, $\beta \in \mathbb{C}$. As before, $\alpha\overline{\Lambda} = \Lambda$, but $\overline{\Lambda} = \Lambda$, so $\alpha = \pm 1$. If $\alpha = -1$, replacing Λ by $i\Lambda$ preserves its reality, and changes α to 1.

With $\alpha = 1$, the condition that I is involutive is that $\beta + \overline{\beta} \in \Lambda$. By the change of coordinate $u \mapsto u - i\ \mathrm{Im}\,\beta$, it can be assumed that $\beta \in \mathbb{R}$. Then the involutive condition is that $2\beta \in \Lambda$. If $\beta \in \Lambda$ then 0 is a fixed point of I. Hence $\beta \equiv \omega_1$ (rectangle) or $\beta = \omega_1 + \omega_3$ (rhombus). In the latter case, ω_1 is a fixed point of I, so the only admissible case is the rectangle, with $I(u) = \overline{u} + \omega_1$.

Proof of (iii). The compatibility condition in Theorem 8 demands that $I^*I^*(s) = -s$ for any section s of the spin structure. A computation shows that this condition is met only for the two spin structures named.

Proof of (iv). Let γ_1 and γ_3 be respectively the closed curves $t \mapsto \omega_1 t/|\omega_1| + c_1$ and $t \mapsto \omega_3 t/|\omega_3| + c_2$, $(0 \le t \le 2)$, where c_1, $c_2 \in \mathbb{C}$ are chosen so that the curves do not pass through any ends. Then $I(\gamma_1) = \gamma_1$, $I(\gamma_3) = -\gamma_3$. The period conditions are

$$\int_{\gamma_k} s_1^2 = \overline{\int_{\gamma_k} s_2^2} \quad \text{and} \quad \int_{\gamma_k} s_1 s_2 \in i\mathbb{R} \quad (k = 1, 3).$$

With I as above, under the double-cover assumption

$$(s_1, s_2) = \pm(i\overline{I^* s_2}, -i\overline{I^* s_1}),$$

we have

$$\int_{\gamma_3} s_1^2 = \int_{\gamma_3} -\overline{I^* s_2^2} = -\overline{\int_{I(\gamma_3)} s_2^2} = \overline{\int_{\gamma_1} s_2^2}$$

$$\int_{\gamma_3} s_1 s_2 = \int_{\gamma_3} \overline{I^* s_1 s_2} = \overline{\int_{I(\gamma_3)} s_1 s_2} = -\overline{\int_{\gamma_3} s_1 s_2},$$

so the period conditions are automatically satisfied for $k = 3$. Moreover, we also have

$$\int_{\gamma_1} s_1^2 = \int_{\gamma_1} -\overline{I^* s_2^2} = -\overline{\int_{I(\gamma_1)} s_2^2} = -\overline{\int_{\gamma_1} s_2^2}$$

$$\int_{\gamma_1} s_1 s_2 = \int_{\gamma_1} \overline{I^* s_1 s_2} = \overline{\int_{I(\gamma_1)} s_1 s_2} = \overline{\int_{\gamma_1} s_1 s_2}$$

and the first two period conditions (9) become

$$\int_{\gamma_1} s_1^2 = 0 \quad \text{and} \quad \int_{\gamma_1} s_1 s_2 = 0$$

(this amounts to three real conditions because, under the above assumption, the second integral is automatically real). $\square$

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