TRIUNDULOIDS:
EMBEDDED CONSTANT MEAN CURVATURE SURFACES WITH THREE ENDS AND GENUS ZERO

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Abstract. In 1841, Delaunay constructed embedded surfaces of revolution with constant mean curvature (CMC). These *unduloids* have genus zero and are now known to be the only embedded CMC surfaces with two ends. Here, we construct the complete family of CMC surfaces with three ends and genus zero; they are classified by their asymptotic necksizes. We work in a class slightly more general than embedded surfaces, namely immersed surfaces which bound an immersed three-manifold, introduced by Alexandrov.

Introduction

Surfaces which minimize area under a volume constraint have constant mean curvature; this condition can be expressed as a nonlinear partial differential equation. We are interested in complete surfaces properly embedded in $\mathbb{R}^3$ which have nonzero constant mean curvature (CMC). By a rescaling we assume the mean curvature equals 1. Alexandrov [A1] showed that the unit sphere is the only compact embedded CMC surface. Thus the simplest nontrivial case to consider is surfaces of *finite topology*. These are homeomorphic to a compact surface Σ of genus g with a finite number k of points removed. A neighborhood of each of these punctures is called an *end* of the surface.

The *unduloids*, CMC surfaces of revolution described by Delaunay [D], are genus-zero examples with two ends. The (periodic) generating curve for each unduloid solves an ordinary differential equation. Up to rigid motion, the entire family is parametrized by the unduloid *necksize*, which measures the circumference of the necks (shortest closed geodesics) in the surface. This necksize ranges from zero (at the singular chain of spheres) to π (at the cylinder).

Many new CMC surfaces of finite topology have been constructed in recent decades. But embeddedness has been proven only for certain countable families among these, with genus zero and up to 12 ends, constructed by Kapouleas [Kp]. In fact, embeddedness is not at all straightforward, since continuous families of embedded CMC surfaces may develop self-intersections on compact sets.

Therefore, it is reasonable to consider a more general class of surfaces, as introduced by Alexandrov [A2]. A compact immersed surface is *Alexandrov embedded* if it bounds a compact immersed three-manifold; this is the natural class of compact surfaces to which Alexandrov reflection applies. The correct extension to finite topology surfaces is more subtle, but again is chosen to ensure that a generalized Alexandrov reflection will work.

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Definition. A CMC surface M of finite topology is *Alexandrov embedded* if M is properly immersed, if each end of M is embedded, and if there exists a compact 3-manifold W with boundary $\partial W = \Sigma$, and a proper immersion $F\colon W \smallsetminus \{q_1, \dots, q_k\} \to \mathbb{R}^3$ whose boundary restriction $f\colon \Sigma \smallsetminus \{q_1, \dots, q_k\} \to \mathbb{R}^3$ parametrizes M. Moreover, we assume that the mean-curvature normal of M points into W.

We define a *triunduloid* to be an Alexandrov embedded CMC surface with genus zero and three ends. These are the simplest nontrivial CMC surfaces. Indeed, there are no Alexandrov embedded CMC surfaces with one end [Mk], and the unduloids are the only examples with two ends [KKS]. Certain triunduloids, with small necksizes or high symmetry, were previously known to exist, due to work by Kapouleas [Kp], Grosse-Brauckmann [G], and Mazzeo and Pacard [MP].

Unlike more general finite topology CMC surfaces, triunduloids have a special symmetry property. By a theorem of Meeks [Mk], any triunduloid is contained in a half-space. Korevaar, Kusner and Solomon [KKS] extend the Alexandrov reflection method to Alexandrov embedded CMC surfaces contained in a half-space. It follows that any such surface M is *Alexandrov-symmetric* in the following sense: it has a reflection plane P such that $M \smallsetminus P$ consists of two connected components $M^\pm$ whose Gauss images $\nu(M^\pm)$ are contained in the open hemispheres $\mathbb{S}^2_\pm$; moreover $\nu(M \cap P)$ is contained in the equator. Together with the asymptotics result of [KKS] this gives for triunduloids (of any genus):

Theorem 0.1 ([KKS]). *A triunduloid is contained in a slab and is Alexandrov-symmetric. Each of its ends is exponentially asymptotic to a Delaunay unduloid, and thus has an asymptotic axis and necksize $n \in (0, \pi]$.* □

Our main theorem below classifies all triunduloids. We identify two surfaces when they differ by a rigid motion; however, to avoid orbifold points in the moduli space, we first label the ends.

Definition. The *moduli space $\mathcal{M}$ of triunduloids* consists of all proper Alexandrov embedded CMC immersions of $\mathbb{S}^2 \smallsetminus \{q_1, q_2, q_3\}$ in $\mathbb{R}^3$, modulo diffeomorphisms of the domain fixing the points q_i, and modulo orientation-preserving isometries of $\mathbb{R}^3$. Uniform convergence in Hausdorff distance on compact subsets of $\mathbb{R}^3$ defines the topology on $\mathcal{M}$.

Our classification says that this space $\mathcal{M}$ is homeomorphic to the three-ball of triples of points on the two-sphere, which we make precise as follows:

Definition. The space $\mathcal{T}$ of *triples* consists of all ordered triples of distinct points in $\mathbb{S}^2$, up to rotation; that is, $\mathcal{T}$ is the open three-ball

$$\mathcal{T} := \{(p_1, p_2, p_3) \in (\mathbb{S}^2)^3, p_1 \neq p_2 \neq p_3 \neq p_1\}/\mathsf{SO}(3).$$

To check explicitly that this is a ball, we may rotate an ordered triple until it lies clockwise along a common latitude, and the first point is on a fixed longitude. Then the common latitude and the two other longitudes give three coordinates.

Main Theorem. *There is a homeomorphism Ψ from the moduli space $\mathcal{M}$ of triunduloids to the open three-ball $\mathcal{T}$ of triples in the sphere $\mathbb{S}^2$. The necksizes of a triunduloid $M \in \mathcal{M}$ are the spherical distances of the triple $\Psi(M)$.*

Thus spherical trigonometry can be used to give information on triunduloids, such as the following (where the last statement follows the fact that cylinders have necksize π.)

Corollary. *Three numbers n_1, n_2, n_3 in $(0, \pi]$ can be the necksizes of a triunduloid if and only if they satisfy the spherical triangle inequalities*

$$n_1 \leq n_2 + n_3, \quad n_2 \leq n_3 + n_1, \quad n_3 \leq n_1 + n_2,$$
$$n_1 + n_2 + n_3 \leq 2\pi.$$

In particular, at most one end of a triunduloid can be asymptotic to a cylinder.

When all these inequalities are strict, a triple in $\mathcal{T}$ has a distinct mirror image, so in this case there are exactly two triunduloids with the same three necksizes. The force balancing formula (see Section 4) leads to explicit trigonometric formulas relating the necksizes to the *angles* made by the asymptotic axes of the ends (see [GKS1]). However, we do not have equally good control on the *phase* of the ends, meaning the positions of the asymptotic necks along the axis.

We now outline the proof of our theorem (see also [GKS2]). The main steps are to define the map Ψ from triunduloids to spherical triples, and then prove injectivity, properness, and surjectivity.

In Section 2 we define Ψ. We make use of Alexandrov symmetry to decompose a triunduloid in $\mathcal{M}$ into two halves. Each half is simply connected, so Lawson's conjugate cousin construction [La] gives a minimal cousin in the three-sphere $\mathbb{S}^3$. The three boundary components of the cousin are each tangent to the same Hopf fibration [Kr], and hence they project under the Hopf map to the three points of a spherical triple in $\mathcal{T}$.

Our approach to conjugation, explained in Section 1, is different from Lawson's. It shows the cousin itself Hopf-projects to an immersed disk in $\mathbb{S}^2$. The spherical metric induced on this disk depends only on the triple in $\mathcal{T}$. We regard the cousin as a section of the Hopf circle bundle over this disk, and so any other triunduloid with the same triple defines a section of the same bundle. Hence the maximum principle applied in the bundle proves the injectivity of Ψ; this is the main result of Section 3.

Knowing the injectivity of Ψ, the surjectivity proof amounts to an existence proof for triunduloids via a continuity method, applied in the noncompact space $\mathcal{T}$. In Section 4 we prove Ψ is proper; this uses the area estimate and extends the curvature estimate from [KK]. To see $\Psi(\mathcal{M})$ is open in $\mathcal{T}$ we depend on the fact [KMP] that the moduli space $\mathcal{M}$ of triunduloids is locally a real analytic variety. This variety has dimension three as is shown by using a nondegenerate minimal trinoid [MR] in a construction of Mazzeo and Pacard [MP]. A topological argument, given in Section 5, shows that a proper injective map from the three-variety $\mathcal{M}$ to the connected three-manifold $\mathcal{T}$ must be surjective and, in fact, a homeomorphism.

Our theorem raises the question whether all triunduloids are nondegenerate. This would be the prerequisite to using triunduloids in various gluing constructions. Intuitively, we expect that triunduloids can be glued together to construct CMC surfaces of any finite topology.

Our results generalize to the case of *coplanar k-unduloids* of genus 0, meaning Alexandrov embedded CMC surfaces with k ends whose asymptotic axes lie in a common plane. Since this case is technically more involved, we devote a forthcoming paper to it.

Cosín and Ros [CR] adapted our approach to classify coplanar minimal k-noids in $\mathbb{R}^3$; more recently, Rosenberg has announced a result similar to ours for mean-curvature-1 trinoids in hyperbolic space.

$MC > 1$ in hyp?

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1. Conjugate cousins

1.1. **Conjugate minimal surfaces.** A simply connected minimal surface M in $\mathbb{R}^3$ has, as is well-known, a conjugate minimal surface $\tilde{M}$. These surfaces are isometric, and at corresponding points they have the same tangent plane; that is the surfaces share the same Gauss map. However, tangent vectors are rotated by a right angle under the isometry.

In terms of a parameterization $f : \Omega \to \mathbb{R}^3$ of M (where $\Omega \subset \mathbb{R}^2$) and the corresponding parameterization $\tilde{f}$ of $\tilde{M}$, conjugation can be described as follows. Since f and $\tilde{f}$ are isometric, the induce the same pullback metric on Ω; we let J denote the rotation by $90°$ with respect to this metric. Then the parameterizations are related by the first-order differential system

$$d\tilde{f} = df \circ J. \tag{1}$$

Note that since the rotation in the tangent plane in $\mathbb{R}^3$ can be effected by a cross product with the normal vector, we could also write this equation as $d\tilde{f} = n \times df$, without reference to the pullback metric on Ω.

If f is conformal, J is the ordinary rotation in $\mathbb{R}^2$, and (1) becomes the Cauchy-Riemann system

$$\tilde{f}_x = f_y, \qquad \tilde{f}_y = -f_x.$$

That is, when a minimal surface M is given a conformal parameterization, its coordinate functions are harmonic, and those of $\tilde{M}$ are the conjugate harmonic functions.

Symmetry curves on conjugate surfaces have a striking relationship. If $f \circ \gamma$ is any curve in M then (1) implies that the *cornormal* vector $\eta := df(J\gamma')$ equals the tangent vector $d\tilde{f}(\gamma')$ of the conjugate curve in $\tilde{M}$. Thus $f \circ \gamma$ has constant conormal $u \in \mathbb{S}^2$, if and only if $\tilde{f} \circ \gamma$ is contained in a straight line in direction $u \in \mathbb{S}^2$; that is, an *arc of planar reflection* in M corresponds to an *arc of rotational symmetry* in $\tilde{M}$.

According to Bonnet's fundamental theorem, a surface in space is determined up to rigid motion by second-order information: its metric and its shape operator. This leads to an equivalent, second-order description of conjugation: M and $\tilde{M}$ are isometric, and their shape operators are related by $J \circ \tilde{S} = S$. The Gauss and Codazzi equations, the intergrability conditions in Bonnet's theorem, confirm that the conjugate surface $\tilde{M}$ exists exactly when M is minimal.

1.2. **Defining conjugate cousins.** Lawson [La] described a similar notion of *conjugate cousin* surfaces, where M is a CMC surface in $\mathbb{R}^3$ and $\tilde{M}$ is an isometric minimal surface in $\mathbb{S}^3$. He gave a second-order description, where the shape operators are related by $J \circ \tilde{S} = S - \mathrm{Id}$. Again, the integrability conditions in Bonnet's theorem prove existence of $\tilde{M}$ exactly when M is CMC and, conversely, existence of M exactly when $\tilde{M}$ is minimal [La, Kr, G]. It is the Gauss equation which shifts the unit mean curvature of $M \subset \mathbb{R}^3$ to ambient sectional curvature of $\mathbb{S}^3 \supset \tilde{M}$.

We want to give a description of conjugate cousins via a first-order system of ordinary differential equations. When properly interpreted, it says again that the surfaces share a common normal, and tangent vectors rotate by a right angle. This description seems to have been first mentioned in [GP], while the first proof (in a different context) appears in [O]. Karcher's earlier work [Kr] can be regarded as a first-order description of the symmetry curves alone.

System (1) for minimal surfaces in $\mathbb{R}^3$ equates tangent vectors at two different points of $\mathbb{R}^3$, implicitly making use of parallel translation. In the case of conjugate cousins, to compare tangent vectors to $\mathbb{R}^3$ and $\mathbb{S}^3$, we use the Lie group structure of $\mathbb{S}^3$. First we fix an identification of $\mathbb{R}^3$ with the Lie algebra $T_1\mathbb{S}^3$, and then we identify this with the tangent space at any point $p \in \mathbb{S}^3$ by left translation $L_p : T_1\mathbb{S}^3 \to T_p\mathbb{S}^3$. Making this explicit, we generalize (1) to

$$d\tilde{f} = L_{\tilde{f}}\,(df \circ J). \tag{2}$$

That is, given a tangent vector to the CMC surface $f\colon \Omega \to \mathbb{R}^3$, we first rotate it by 90 degrees within the tangent plane. This rotated vector is an element of $T_{f(x)}\mathbb{R}^3 \cong \mathbb{R}^3 \cong T_1\mathbb{S}^3$, and we left-translate it by $\tilde{f}(x)$ to lie in $T_{\tilde{f}(x)}\mathbb{S}^3$.

1.3. **Quaternions.** The Lie group $\mathbb{S}^3$ is a matrix group: the set of unit quaternions (or equivalently $\mathsf{SU}(2)$). In this framework, points in $\mathbb{S}^3$ and tangent vectors to $\mathbb{S}^3$ are both represented by quaternions, and left translation is simply multiplication. We identify $\mathbb{R}^3$ with $T_1\mathbb{S}^3 = \mathrm{Im}\,\mathbb{H}$, the imaginary quaternions. Equation (2) then becomes the quaternion-valued equation

$$\tilde{f}^{-1}\,d\tilde{f} = df \circ J.$$

Again, if f (or equivalently $\tilde{f}$) is conformal, we can write this as

$$\tilde{f}^{-1}\,\tilde{f}_x = f_y, \qquad \tilde{f}^{-1}\,\tilde{f}_y = -f_x. \tag{3}$$

In order to investigate the integrability of this system, let us now briefly review some of the geometry of $\mathbb{S}^3 \subset \mathbb{H} = \mathbb{R}^4$ in quaternionic notation.

For imaginary quaternions, $p, q \in \mathrm{Im}\,\mathbb{H}$, we can express the quaternionic product in terms of the ordinary dot and cross products: $\mathrm{Re}(pq) = -\langle p, q\rangle \in \mathrm{Re}\,\mathbb{H} = \mathbb{R}$ and $\mathrm{Im}(pq) = p \times q \in \mathrm{Im}\,\mathbb{H} = \mathbb{R}^3$, or $pq = -\langle p, q\rangle + p \times q$. In particular, imaginary p and q commute if and only if they are parallel, and anticommute if and only if they are orthogonal. Therefore, conformality for a map $f\colon \Omega \to \operatorname{Im}\mathbb{H}$ can be expressed as

$$|f_x|^2 = |f_y|^2, \qquad f_x f_y = -f_y f_x. \tag{4}$$

Given a surface in a oriented three-dimensional space, a choice of normal is equivalent to a choice of orientation for the surface. In particular, the (interior) *mean curvature normal* ν gives a CMC surface a unique conventional orientation, while a minimal surface can oriented either way. Once we orient the surface, we will always use oriented parameterizations, such that $f_x \times f_y$ is a positive multiple of ν. The $+90$ degree rotation J on the domain is then oriented in the usual counter-clockwise manner, so that (for any $X \in T\Omega$) $(df(X), df(JX), \nu)$ is oriented in space.

With the above conventions, the mean curvature H of a conformally parameterized surface $f : \Omega \to \mathbb{R}^3$ can be measured by $\Delta f = 2H f_x \times f_y$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the usual Laplacian on Ω. Expressing the cross product quaternionically, we see that f is CMC (meaning that $H \equiv 1$) if and only if

$$\Delta f = 2 f_x f_y. \tag{5}$$

Similarly, a conformal $\tilde{f}\colon \Omega \to \mathbb{S}^3$ is minimal if and only if it is harmonic:

$$\Delta \tilde{f} = -\tilde{f}|d\tilde{f}|^2, \tag{6}$$

where $|d\tilde{f}|^2 = |\tilde{f}_x|^2 + |\tilde{f}_y|^2$.

1.4. **Existence of cousins.** We are now ready to prove the theorem giving the existence of isometric conjugate cousin surfaces $M \subset \mathbb{R}^3$ and $\tilde{M} \subset \mathbb{S}^3$. We will find that (5) is the integrability condition for $\tilde{f}$, and (6) is that for f, so that cousins exist exactly when f is CMC or $\tilde{f}$ is minimal.

Theorem 1.1. *Let Ω be a simply connected domain in $\mathbb{R}^2$. Then, for each immersion $f\colon \Omega \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ of constant mean curvature 1, there is an isometric minimal immersion $\tilde{f}\colon \Omega \to \mathbb{S}^3$, determined uniquely up to left translation, satisfying*

$$d\tilde{f} = \tilde{f}\, df \circ J. \tag{7}$$

Conversely, given an oriented minimal $\tilde{f}$, there is a CMC immersion f, unique up to translation, satisfying (7). In either case, the surface normals agree, in the sense that $\tilde{\nu} = \tilde{f}\nu$.

Proof. The results are invariant under orientation-preserving diffeomorphisms of Ω. Thus it is sufficient to prove the theorem assuming a conformal parametrization.

We use two facts about any solutions f and $\tilde{f}$ of (7), which are straightforward to check. First, $|\tilde{f}|$ is constant if and only if $\operatorname{Re} f$ is constant; thus a solution with initial value in $\mathbb{S}^3$ or $\operatorname{Im}\mathbb{H}$ will stay within this same subset of $\mathbb{H}$. Second, the metrics induced by f and $\tilde{f}$ are conformal, with lengths differing by the factor $|\tilde{f}|$. This confirms that M and $\tilde{M}$ are isometric. (Geometrically, these facts were clear from our description in the previous subsection, where tangent vectors to the two surfaces were defined to have the same lengths, and to lie within $T\mathbb{S}^3$ or $T\mathbb{R}^3$.)

Now let $f\colon \Omega \to \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ be conformal, and set $\alpha = df \circ J = f_y\,dx - f_x\,dy$, so that equation (7) becomes $d\tilde f = \tilde f \alpha$. The integrability condition is then

$$0 = d(\tilde f\alpha) = d\tilde f \wedge \alpha + \tilde f d\alpha = \tilde f(\alpha\wedge\alpha + d\alpha).$$

But we calculate $d\alpha = -\Delta f\ dx\wedge dy$, and $\alpha\wedge\alpha = f_x f_y\,dx\wedge dy + f_y f_x\,dy\wedge dx = 2f_x f_y\,dx\wedge dy$, using (4). Thus the integrability condition is precisely (5), so a solution $\tilde f$ exists exactly when f has constant mean curvature 1, and it is determined up to left translation.

Conversely, given a conformal $\tilde f\colon \Omega \to \mathbb{S}^3$, and setting $\beta = -d\tilde f \circ J$, we can write equation (7) as $df = \tilde f^{-1}\beta$. Thus the integrability condition is

$$0 = d(\tilde f^{-1}\beta) = d\tilde f^{-1}\wedge\beta + \tilde f^{-1}d\beta = -\tilde f^{-1}d\tilde f\tilde f^{-1}\wedge\beta + \tilde f^{-1}d\beta.$$

From (3) we have $(\tilde f^{-1}\tilde f_x)^2 = f_y^2$. Moreover, f_y is pure imaginary, and so $f_y^2 = -|f_y|^2$, which also equals $-|\tilde f_y|^2$ as the cousins are isometric. Similarly, $(\tilde f^{-1}\tilde f_y)^2 = -|f_x|^2$, so $(\tilde f^{-1}\tilde f_x)^2 + (\tilde f^{-1}\tilde f_y)^2 = -|d\tilde f|^2$, and the first term in the integrability condition becomes $|d\tilde f|^2\,dx\wedge dy$. The second term is $\tilde f^{-1}\Delta\tilde f\,dx\wedge dy$. Comparing with (6), we see that our system is integrable for f exactly when $\tilde f$ is minimal. □

Left-translating a unit vector $u \in \mathbb{S}^2 \subset \operatorname{Im}\mathbb{H}$ gives us a left-invariant vector field $p \mapsto pu$ on $\mathbb{S}^3$, called the *u-Hopf field.* Its integral curve through $p \in \mathbb{S}^3$ is a great circle (a *u-Hopf circle*), parameterized by pe^{tu}; here we use the notation $e^{tu} := \cos t + u\sin t$ for the great circle through 1 and v. The *u-Hopf projection* $\Pi_u\colon \mathbb{S}^3 \to \mathbb{S}^2$ is defined by $\Pi_u(p) = pu\overline{p}$. This projection is a fibration; its fibers are the u-Hopf circles, since $\Pi_u(pe^{tu}) = pe^{tu}ue^{-tu}\overline{p} = \Pi_u(p)$.

Since left translation is an isometry for the metric on $\mathbb{S}^3$ we conclude from $\tilde\nu = \tilde f\nu$ that $\langle\tilde\nu, \tilde f u\rangle_{\mathbb{S}^3} = \langle\nu, v\rangle_{\mathbb{R}^3}$ on Ω. Consequently:

Corollary 1.2. *A CMC surface M is transverse to the foliation of $\mathbb{R}^3$ by lines in the u direction if and only if its minimal cousin $\tilde M$ is transverse to the foliation of $\mathbb{S}^3$ by u-Hopf circles.*

If a surface $\tilde M \subset \mathbb{S}^3$ is transversal to the u-Hopf fibration then $\Pi_u(\tilde M)$ immerses into $\mathbb{S}^2$; we will apply the Corollary in this form.

Our first-order description of conjugate cousins does in fact agree with Lawson's original second-order description [La]:

Proposition 1.3. *The shape operators S of f and $\tilde S$ of $\tilde f$ are related by $J\circ\tilde S = S - \mathrm{Id}$.*

Proof. By definition of the shape operator, $d\tilde f\circ\tilde S = -d\tilde\nu = -d\tilde f\,\nu - \tilde f\,d\nu$. Substituting $d\tilde f = \tilde f\,df\circ J$ and $d\nu = -df\circ S$, we get

$$df\circ J\circ\tilde S(\cdot) = -\big(df\circ J(\cdot)\big)\,\nu + df\circ S(\cdot).$$

The first term can be simplified: $df\circ J$ takes values in the tangent plane to M; since ν is the normal vector, multiplication by ν simply rotates by 90°, and we see $\big(df\circ J(\cdot)\big)\nu = \big(df\circ J(\cdot)\big)\times\nu = df(\cdot)$. Therefore, we get $J\circ\tilde S = -Id + S$ as desired. □

As on conjugate minimal surfaces, symmetry curves on cousin surfaces are related. This time, a curve $f \circ \gamma$ in M with constant conormal v gives a curve $\tilde{f} \circ \gamma$ in $\tilde{M}$ with tangent vector in the constant Hopf field $\tilde{f}v$. We thus get:

Theorem 1.4. *If M and $\tilde{M}$ are conjugate cousins, then $f \circ \gamma$ is a curve of planar reflection in M for a plane with unit normal $v \in \mathbb{S}^2$ if and only if the curve $\tilde{f} \circ \gamma$ in $\tilde{M}$ is contained in a u-Hopf circle in $\mathbb{S}^3$.*

1.5. **Unduloids.** The simplest example for cousin surfaces is the unit sphere $\mathbb{S}^2 = \mathbb{R}^3 \cap \mathbb{S}^3$ which is its own conjugate cousin. Indeed, if f conformally parametrizes $\mathbb{S}^2$ then $f\, df \circ J = f \times df \circ J = df$ and so $\tilde{f} = f$.

Let us now consider an unduloid U of necksize $n \in (0, \pi]$. We will show that its cousin $\tilde{U}$ is the *spherical helicoid* parametrized by

$$\psi\colon \mathbb{R}^2 \to \mathbb{S}^3, \qquad \psi(x,y) := e^{-ny\mathbf{i}} e^{xu(y)}, \qquad u(y) := e^{2\pi y \mathbf{i}}\mathbf{j}, \tag{8}$$

written with respect to the standard basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{H} = \mathbb{R}^4$.

The surface ψ is ruled by great circles: For any fixed y, the x-coordinate line $x \mapsto \psi(x,y)$ traces out the $u(y)$-Hopf circle through $e^{-ny\mathbf{i}}$. By Theorem 1.4 the cousin CMC surface is foliated by arcs of planar reflection; the normals to the mirror planes are the tangent Hopf fields $u(y) = \mathbf{j}\cos(2\pi y) + \mathbf{k}\sin(2\pi y)$ of these circles. Since $u(y)$ rotates through all vectors perpendicular to $\mathbf{i}$, the cousin is a surface of revolution around the x-axis.

The rulings are orthogonal to the *axis* of the helicoid, the $\mathbf{i}$-Hopf circle $y \mapsto \psi(0,y) = e^{-ny\mathbf{i}}$ through 1. In fact, the helicoid has another axis, $y \mapsto \psi(\frac{\pi}{2}, y) = e^{-ny\mathbf{i}}e^{2\pi y\mathbf{i}}\mathbf{j} = \mathbf{j}e^{-(2\pi-n)y\mathbf{i}}$, along the $-\mathbf{i}$-Hopf circle through $\mathbf{j}$, the polar circle of the original axis. Each ruling intersects both axes; its tangent vector at the first axis equals its position on the polar axis: $\frac{\partial\psi}{\partial x}(0,y) = \psi(\frac{\pi}{2}, y)$. Since the polar axis is parametrized with constant speed, the tangent vectors rotate with constant speed, thus justifying the name spherical helicoid.

Lemma 1.5. *If we fix $n \in (0, \pi]$, the spherical helicoid $\psi(x,y)$, given by (8) for $x \in \mathbb{R}$, $y \in (-\frac{1}{4}, \frac{1}{4})$, parametrizes the cousin $\widetilde{U^+}$ of a half unduloid U^+ with necksize n, whose boundary ∂U^+ is contained in the xy-plane. Moreover, the Hopf projection $\Pi_{\mathbf{k}}(\widetilde{U^+})$ (or more precisely $\Pi_{\mathbf{k}} \circ \psi$) is an immersion into the two-sphere $\mathbb{S}^2$ such that the boundary $\Pi_{\mathbf{k}}(\partial\widetilde{U^+})$ consists of two points at spherical distance n.*

Proof. The cousin of ψ is a CMC surface of revolution with meridians contained in mirror planes with normal $v(y)$. So an upper half is indeed conjugate to a domain of parameterization $y \in (-\frac{1}{4}, \frac{1}{4})$. The axes of the helicoid give circles of planar reflection on the CMC surface, contained in planes with normal $\mathbf{i}$. In particular, the axis gives a circle with circumference n, while the the polar axis gives a circle with larger circumference $2\pi - n$.

Delaunay characterized the CMC surfaces of revolution [D]: besides the sphere, these are embedded unduloids and immersed nodoids. For unduloids, neck and bulge circumference sum to 2π, while for nodoids their difference is 2π; therefore ψ is the cousin of an unduloid. We remark that the nodoids are cousins of the helicoids $e^{-ny\mathbf{i}}e^{-xu(y)}$ with $n > 0$; their two different symmetry circles have opposite conormals.

It follows that U^+ is transversal to the z-lines, and so $\Pi_{\mathbf{k}}(\widetilde{U^+})$ is an immersion by Corollary 1.2. Moreover, the bounding rulings $\psi(x, \pm\frac{1}{4}) = e^{(\mp n/4)\mathbf{i}} e^{x\mathbf{k}}$ are $\mathbf{k}$-Hopf circles through $e^{(\mp n/4)\mathbf{i}}$ with $\mathbf{k}$-Hopf projections $e^{(\mp n/4)\mathbf{i}}\mathbf{k}e^{(\pm n/4\mathbf{i})} = \mathbf{k}e^{(\pm n/2)\mathbf{i}}$. Indeed, these two points in $\mathbb{S}^2$ have spherical distance n.

□

2. The classifying map

Theorem 0.1 asserts that a triunduloid is Alexandrov symmetric with respect to a plane P. Remembering that the elements of our triunduloid moduli space $\mathcal{M}$ are equivalence classes of triunduloids up to rigid motion, we can assume the triunduloid M representing a class has been rotated so that P is the xy-plane in $\mathbb{R}^3$, and so that the ends of M, in order of increasing label, have axis vectors in counter-clockwise order in P. For each class in $\mathcal{M}$, this determines M uniquely up to horizontal translation. We call $M^+ := M \cap \{z > 0\}$ the *upper half* of M.

Since M has genus zero, M^+ is simply connected. Hence Theorem 1.1 gives a conjugate cousin $\widetilde{M^+} \subset \mathbb{S}^3$, which is determined uniquely up to left translation.

Since ∂M^+ consists of three arcs of reflection, all contained in the plane P with normal $\mathbf{k} \in \mathbb{S}^2$, it follows from Theorem 1.4 that $\widetilde{M^+}$ is bounded by three labelled arcs of $\mathbf{k}$-Hopf circles. Thus $\Pi_{\mathbf{k}}(\partial\widetilde{M^+})$ consists of three points in $\mathbb{S}^2$. We want to use them to define our map $\Psi\colon \mathcal{M} \to \mathcal{T}$, so we need to check that the triple is well-defined up to rotation, and that the three points are distinct.

But $\widetilde{M^+}$, and hence its boundary, is well-defined up to left translation by some $a \in \mathbb{S}^3$, and $\Pi_{\mathbf{k}}(ap) = a\Pi_{\mathbf{k}}(p)\overline{a}$. Since left and right translation in $\mathbb{S}^3$ are both given by rotations, this composition is also a rotation.

We show the triple is distinct by proving the second part of our Main Theorem.

Theorem 2.1. *The spherical distances between the points of the triple $\Pi_{\mathbf{k}}(\partial\widetilde{M^+})$ are the necksizes of the triunduloid M.*

Proof. Let E_i be the i^{th} end of M, that is, any properly embedded neighborhood of $q_i \in \Sigma$, and let $E_i^+ = E_i \cap M^+$. Theorem 0.1 gives that E_i is exponentially asymptotic to a Delaunay unduloid U of necksize n_i. Exponential convergence is preserved by (7) and so the bounding rays of $\widetilde{E_i^+}$ are asymptotic to the great circles bounding the helicoid $\widetilde{U^+}$. However, by Theorem 1.4 the two rays in the boundary of $\widetilde{E_i^+}$ are themselves great circle rays. Thus they must cover the great circles bounding the cousin of the asymptotic unduloid and the claim follows from Lemma 1.5. □

This completes the proof that our *classifying map*

$$\Psi\colon \mathcal{M} \to \mathcal{T}, \qquad \Psi(M) := \Pi_{\mathbf{k}}(\partial\widetilde{M^+}) \in \mathcal{T},$$

is well-defined. Two triunduloids which are close in Hausdorff distance on compact sets are also C^1 (or C^∞) close. Thus, up to left translation in $\mathbb{S}^3$, the cousin surfaces are again close on compact subsets, and so is the Hopf projection of their boundary. We conclude that Ψ is continuous.

Remark. We could equally well define Ψ on a triunduloid M with arbitrary mirror plane P, by using the Hopf projection in the direction of the normal to P. Note that Hopf projection is equivariant with respect to rotations of $\mathbb{S}^3 \subset \mathbb{H}$ which preserve $\mathbb{S}^2 \subset \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$: if A is such a rotation, realized through conjugation by a unit quaternion a, then

$$\Pi_{Au}(Ap) = (ap\overline{a})(au\overline{a})\overline{(ap\overline{a})} = a(pu\overline{p})\overline{a} = A(\Pi_u(p)).$$

Thus surfaces representing the same class in $\mathcal{M}$ have Hopf images related by a rotation, and indeed give the same triple in $\mathcal{T}$.

3. Injectivity

3.1. **Metrics induced by the Hopf projection of a cousin half.** To prove injectivity of Ψ we consider the Hopf projection of the cousin disk $\widetilde{M^+}$, rather than of its boundary alone. As before, M is chosen (uniquely up to horizontal translation) so that the xy-plane is the Alexandrov mirror plane. Then

$$\Phi(M) := \Pi_{\mathbf{k}}(\widetilde{M^+})$$

is well-defined up to rotation of $\mathbb{S}^2$.

Proposition 3.1. *$\Phi(M)$ is an immersed open disk in $\mathbb{S}^2$.*

Proof. By Theorem 0.1 the immersed open disk M^+ is transverse to the vertical lines in $\mathbb{R}^3$. It follows from Corollary 1.2 that $\widetilde{M^+}$ is transverse to the $\mathbf{k}$-Hopf fibration. □

The immersed disk $\Phi(M)$ in $\mathbb{S}^2$ induces a *spherical metric*, i.e., a Riemannian metric of curvature 1, on M^+ (or the diffeomorphic disk $\widetilde{M^+}$). We call this metric g_M. The essential step in proving injectivity of Ψ is to show that the isometry class of this metric depends only on $\Psi(M) \in \mathcal{T}$. The next three subsections carry out the careful analysis of spherical metrics and the asymptotic behavior of g_M needed to prove this.

3.2. **Metrics induced by unduloids.** A spherical metric on an open 2-disk D defines a *developing map* $\varphi : D \to \mathbb{S}^2$, unique up to rotation. If φ is an embedding we call the metric *embeddable*. As a particular case, an embeddable metric may develop onto the complement of a closed great circle arc of length $n \in (0, \pi]$ in $\mathbb{S}^2$; we call such a metric a *slit sphere*. The different isometry classes of slit spheres are characterized by this n, called the *slit length*.

We can *attach* a slit sphere to another one with the same slit length. To do so, we develop the metrics so that the images in $\mathbb{S}^2$ extend one another across a common bounding edge; then we join and pull back to D in an obvious way. The resulting metric has area 8π. Likewise, we can attach infinitely many slit spheres to one side of a slit sphere, provided all slit lengths agree. We call the resulting metric a *ray of slit spheres*; its developing map has infinite degree.

A *half bubble* of an unduloid U is an open subset of the upper half U^+ enclosed by two semicircles around consecutive necks. In the case of the cylinder we let a half bubble be that region enclosed by any two semicircles at distance π.

Lemma 3.2. *Let U^+ be the upper half of an unduloid of necksize n (with boundary in a horizontal plane as in Lemma 1.5), and let $B^+ \subset U^+$ be a half bubble. Then $\Pi_{\mathbf{k}}(\widetilde{B^+})$ is a slit sphere of slit length n. Consequently, $\Pi_{\mathbf{k}}(\widetilde{U^+})$ is isometric to a slit sphere of slit length n with two attached rays of spheres.*

Proof. We pick a closed semicircle $\gamma \subset \partial B^+$ of length $n/2$. The conjugate arc is contained in an $\mathbf{i}$-Hopf axis of the spherical helicoid ψ from Lemma 1.5. It is parametrized as $\tilde\gamma(t) := \psi(0,t)$ with $t \in (-\frac14, \frac14)$ and also has length $n/2$. Since $\mathbf{i} \perp \mathbf{k}$, the $\mathbf{k}$-Hopf projection maps the $\mathbf{i}$-Hopf circle $\tilde\gamma$ to a great circle in $\mathbb{S}^2$, doubling its length. Hence $e := \Pi_{\mathbf{k}} \circ \tilde\gamma$ is a closed great circle arc of length n in $\mathbb{S}^2$. We claim $\Pi_{\mathbf{k}}$ embeds $\widetilde{B^+}$ onto $\mathbb{S}^2 \smallsetminus e$.

The meridians of the bubble are conjugate to the rulings of ψ, parametrized as $x \mapsto \psi(x,y)$ with $x \in (0,\pi)$. These open arcs of length π are $v(y)$-Hopf, with $v(y) \neq \mathbf{k}$. Therefore, none of the arcs contains two points of the same $\mathbf{k}$-Hopf circle. Consequently, $\Pi_{\mathbf{k}} \circ \tilde\gamma$ is an embedded curve in $\mathbb{S}^2$, limiting at both ends to the same point of e.

Using the property that $\Pi_{\mathbf{k}}$ immerses $\widetilde{B^+}$ (Lemma 1.5), we see that these curves foliate $\mathbb{S}^2 \smallsetminus e$, and so in fact $\Pi_{\mathbf{k}}(\widetilde{B^+})$ is an embedding. □ I don't quite follow this last step –jms

3.3. **Metrics induced by triunduloids.** A *(spherical) triangle metric* on D is one which has a developing map that extends continuously to $\overline{D}$ such that $\mathbb{S}^1$ is mapped homeomorphically onto a geodesic triangle with edge lengths in $(0,\pi]$. Such a metric is a *labelled triangle metric* if the cube roots of unity $q_i = e^{(2\pi i/3)\mathbf{i}}$ (where $i \in \mathbb{Z}/3$) develop onto the vertices of the triangle in $\mathbb{S}^2$, and the open arcs $e_i \subset \mathbb{S}^1$, from q_i to q_{i+1}, to the edges. A neighborhood of q_i in D is then isometric to a spherical geodesic wedge of well-defined angle $\varphi_i > 0$, possibly larger than 2π. Two labelled triangle metrics are *labelled isometric* if the isometry of D extends to a homeomorphism of $\overline{D}$ fixing each q_i.

Slit spheres can be attached to triangle metrics. Indeed, we can attach a slit sphere of slit length $|e_i|$ to the edge e_i, producing a triangle metric with the same edge lengths, but area increased by 4π. We can attach a finite number of slit spheres to each edge, or infinitely many as follows:

Definition. An *extended labelled (spherical) triangle metric* is a spherical metric on D, obtained by attaching three rays of spheres to a labelled triangle metric. That is, there are three disjoint open sets $R_i \subset D$, containing e_i in their closure, each of which is isometric to a ray of spheres in such a way that any curve approaching a point of e_i has infinite length.

Metrically, M^+ is the union of a bounded piece and three ends. In the following lemma, we choose a specific geometric decomposition, by picking the arcs of decomposition at the necks of M^+, in a sense made precise by the Hopf projection of $\widetilde{M^+}$.

Lemma 3.3. *The metric g_M, induced on M^+ by the* $\mathbf{k}$*-Hopf projection of $\widetilde{M^+}$, is an extended labelled spherical triangle metric. More precisely, there is a parameterization $f\colon D \to M^+$ such that $\Pi_{\mathbf{k}} \circ \tilde{f}$ induces such a metric.*

Proof. To define f, we require that all curves in D which approach a point of e_i have image diverging to infinity along the i^{th} end of $M^+ \subset \mathbb{R}^3$. Note that conversely curves on M^+ which approach one of the horizontal arcs of planar reflection then have preimage converging to some q_i. The i^{th} half end $E_i^+ \subset M^+$ (as in the proof of Theorem 2.1) is then parametrized by some open neighborhood U_i of e_i.

We claim U_i contains a ray of slit spheres R_i as an isometric subset. Indeed, as E_i is exponentially close in the C^1 norm to an unduloid U of necksize n_i, so is $\widetilde{E_i^+}$ to $\widetilde{U^+}$. Moreover, the bounding rays agree and hence $\Pi_{\mathbf{k}}(\widetilde{E_i^+})$ and $\Pi_{\mathbf{k}}(\widetilde{U^+})$ are asymptotic immersions into $\mathbb{S}^2$ with the same boundary at infinity. Thus they are isometric at infinity, and some neighborhood of e_i is isometric to a ray of spheres $R_i \subset D$. In particular $R_i \subset D$ is bounded by a minimizing geodesic arc and so $M^+ \setminus f(\overline{R_1} \cup \overline{R_2} \cup \overline{R_3})$ carries a labelled triangle metric. $\square$

What does this "boundary at ∞" mean? –jms

3.4. **Combinatorial description of the metrics induced by triunduloids.** Our goal is to analyse the Hopf projection $\Pi_{\mathbf{k}}(\widetilde{M^+})$ in order to decompose M^+ (and hence M) uniquely into bubbles. Our decomposition only depends on $\Psi(M)$, though not in a continuous way. We will need to understand triangle metrics which, as we saw in Lemma 3.3, describe (M^+, g_M) without its ends.

Lemma 3.4. *Each triangle metric is isometric to an embeddable triangle metric with a finite number of attached slit spheres.*

Proof. We choose $\varphi\colon \overline{D} \to \mathbb{S}^2$ as a developing map of the triangle metric so that $\varphi(D)$ is to the left of $\varphi(\mathbb{S}^1)$ (parametrized counter-clockwise as usual). Let us first consider the case that $\varphi(\mathbb{S}^1)$ does not degenerate into a double-covered segment, but is embedded. Then $\mathbb{S}^2 \setminus \varphi(\mathbb{S}^1)$ has two components, on each of which the (finite) number of preimages of φ is constant. If T is the component to the left of $\varphi(\mathbb{S}^1) = \partial T$ it follows that $\varphi^{-1}(\mathbb{S}^2 \setminus \partial T) \subset D$ is the union of $m \in \mathbb{N}$ domains isometric to T and $m-1$ domains isometric to $\mathbb{S}^2 \setminus \overline{T}$. We call these sets *triangles*. The set $\Gamma := \varphi^{-1}(\partial T) \cap D$ forms a (disconnected) graph in D (some of whose edges may extend to the boundary). Its (finite) vertices are two-valent, since φ immerses D.

We claim that each triangle in D has either two or three vertices on $\mathbb{S}^1$. On the contrary, suppose both bounding vertices of an edge $e \in \Gamma$ are contained in the open disk D. Together with two further edges which must be disjoint, e bounds a triangle Δ. Suppose the triangle vertex v opposite to e is also contained in D. Then $\overline{\Delta} \subset D$, and so $\partial\Delta$ forms a connected component of Γ. But no triangle contained in the annulus $D \setminus \overline{\Delta}$ can span the topologically nontrivial loop $\partial\Delta$. Thus we can assume $v \in \mathbb{S}^1$. Again $\partial\Delta$ must bound a triangle in $D \setminus \overline{\Delta}$, for which $\partial\Delta$ is the entire boundary. Again, this is impossible.

If all triangles have exactly two vertices on $\mathbb{S}^1$, then all vertices interior to D map to the same vertex of ∂T. But that means the bounding $\mathbb{S}^1$ must map to just one edge of ∂T, contradicting our assumptions. On the other hand, a triangle with three vertices on $\mathbb{S}^1$ decomposes the bounding $\mathbb{S}^1$ into three components, each of which maps to a single edge of ∂T. Thus there is at most one such triangle.

Consequently there is exactly one triangle Δ in D with three vertices on $\mathbb{S}^1$. It defines the embeddable triangle metric. Using the same arguments as before, we see that each of the three connected components in $D \setminus \overline{\Delta}$ is a chain of an even number of triangles, alternating between type T and $\mathbb{S}^2 \setminus \overline{T}$. We can group adjacent pairs to form sets isometric to slit spheres.

If the three image edges in $\varphi(\mathbb{S}^1)$ are not disjoint then Γ divides D into connected components each of which is isometric to a slit sphere. The above arguments apply similarly and establish that one of these components has the three bounding vertices on $\mathbb{S}^1$. □

For any triunduloid M, Lemma 3.3 assigns a triangle metric to M^+ minus its ends. For this metric, the set Δ is the subset of M^+ enclosed by the three *innermost* necks. The following proof that (M^+, g_M) is determined by $\Psi(M)$ relies on the fact that Δ is determined by $\Psi(M) \in \mathcal{T}$.

Theorem 3.5. *Let M and M' be two triunduloids having the same classifying triple $\Psi(M) = \Psi(M') \in \mathcal{T}$. Then the immersed disks $\left(M^+, g_M\right)$ and $\left({M'}^+, g_{M'}\right)$ are labelled isometric.*

Proof. We use Lemmas 3.3 and 3.4 to decompose the domain D of a parameterization $f\colon D \to M^+$ into $\Delta \cup R_1 \cup R_2 \cup R_3$, where $\Delta \subset D$ is a labelled extended triangle metric which is embeddable, and $R_i \subset D$ are rays of spheres attached to Δ. Let us first consider the case no end of M is cylindrical so that $\Psi(M)$ contains no antipodal points.

Then the three vertices $\Psi(M)$ uniquely determine the three minimizing arcs bounding the triangle $\Pi_{\mathbf{k}} \circ \tilde{f}(\Delta)$. We claim that Δ is isometric to the triangle in $\mathbb{S}^2$ sitting to the left of $\Pi_{\mathbf{k}} \circ \tilde{f}(\partial\Delta)$ when travelling in the order of increasing label. If the complement of $\Pi_{\mathbf{k}} \circ \tilde{f}(\partial\Delta)$ is connected, that is, one triangle arc develops to the union of the two other arcs, this is obvious. Otherwise there are two components and the claim can be checked by an explicit calculation; by the asymptotics of M, it is enough to check it for an unduloid. In fact, all we need to know is that $\Phi(M)$ sits to the same side of $\Pi_{\mathbf{k}} \circ \tilde{f}(\partial\Delta)$ for all $M \in \mathcal{M}$; this, however, follows from our choice of upper half and from the orientation conventions we use.

This seems a little evasive to me. What really happens when a triangle degenerates? –jms

Consequently, given M, M' with $\Psi(M) = \Psi(M') \in \mathcal{T}$, the two embedded triangles Δ and Δ' are isometric. This isometry extends to the entire disks M^+ and $(M')^+$ by mapping spheres in R_i to those in R'_i one by one.

In the case that the i^{th} end of M is cylindrical, we possibly need to change the above decomposition $\Delta \cup R_1 \cup R_2 \cup R_3$. We claim we can always extend Δ at the expense of R_i so that Δ gets area 4π; for such a choice, again $\Psi(M)$ uniquely determines Δ by the same arguments as before. To prove the claim, note that the upper half of a cylinder has a continuous group of symmetries, given by translation. Consequently, the spherical metric induced on a cylindrical end admits a continuous choice of grouping into spheres; we choose one which assigns area 4π to Δ. □

Why do we choose 4π here, not 0 or 2π or whatever? –jms

3.5. **A uniqueness theorem for triunduloid cousins.** It is well-known that in $\mathbb{R}^3$ two minimal graphs over the same domain agree provided they have the same boundary data. Indeed, by a vertical translation, the graphs can be assumed to have a one-sided touching in which case they agree by the maximum principle. The same argument extends to the case of surfaces transverse to vertical lines; we will generalize it to our setting of a cousin disk $\widetilde{M^+} \subset \mathbb{S}^3$ transverse to the $\mathbf{k}$-Hopf fibration.

To achieve a one-sided contact of two cousin disks it will be essential to unwrap the $\mathbf{k}$-Hopf circles of $\mathbb{S}^3$ to lines. For this reason we work in a noncompact ambient manifold E, which, however, depends on M.

Consider a triple $T \in \Psi(\mathcal{M})$ associated to some triunduloid M. According to Theorem 3.5 the extended labelled triangle metric $g_T := g_M$ is well defined even if there are other triunduloids mapping to T.

Now consider $\mathbb{S}^3$ as the $\mathbf{k}$-Hopf bundle over S^2, and pull it back under the developing map $\varphi : D \to \mathbb{S}^2$ to get a circle-bundle over D. Finally, noting that this bundle is topologically trivial, take the universal cover in the fiber direction to get a line bundle $E = E(t)$ over D. Note that at both steps we can pull back the Riemannian metric, so that $E \to D$ is locally isometric to $\mathbb{S}^3 \to \mathbb{S}^2$. There is a fiberwise and isometric action of $\mathbb{R}$ on E, namely the pullback of the $\mathbf{k}$-Hopf flow on $\mathbb{S}^3$; we refer to this as *vertical translation* in E.

For any triunduloid $M \in \mathcal{M}$ with $\Psi(M) = t$, the cousin disk $\widetilde{M^+}$ lifts to a section of $E(T)$. Indeed, by Proposition 3.1, $\widetilde{M^+}$ is transverse to the $\mathbf{k}$-Hopf field and so locally lifts to a section of $E(T)$; but the bundle is trivial and so the lift global. This section is only well-defined up to vertical translation. Conversely, a minimal section of $E(T)$ locally looks like a minimal surface in $\mathbb{S}^3$, and so it has a CMC cousin in $\mathbb{R}^3$.

Now consider any minimal section $\tilde{S}$ of $E(T)$. Though $E(T)$ is not complete, all curves on $\tilde{S}$ approaching a point of $(\mathbb{S}^1 \smallsetminus \{q_1, q_2, q_3\}) \times \mathbb{R}$ have infinite length. Thus the completion of $\tilde{S}$ is given by the three $\mathbf{k}$-lines over q_1, q_2, q_3. So each boundary curve of S is contained in a horizontal plane, and the cousin disk $S \subset \mathbb{R}^3$ has exactly three boundary components. These three planes may not coincide, however, as they do for a triunduloid half. So the difference in height of the two planes containing the boundary curves limiting in end i defines a possibly nonzero *period* $P(i) \in \mathbb{R}$.

Lemma 3.6. *Consider two minimal sections $\widetilde{S}_1$, $\widetilde{S}_2$ of $E(T)$. Suppose the sections intersect in a piecewise smooth curve $\tilde{\gamma}$ of finite length such that there is a one-sided neighbourhood of $\tilde{\gamma}$ in $\widetilde{S}_1$ which does not intersect $\widetilde{S}_2$. If the projection of $\tilde{\gamma}$ has endpoints q_i and q_{i+1} then the i^{th} periods $P_j(i)$ of the surfaces S_j satisfy $P_1(i) \neq P_2(i)$.*

I think this proof needs some work. In particular, $\mathbf{k}$ does not mean the $\mathbf{k}$-Hopf field. –jms

Proof. Since $\tilde\gamma$ has finite length we assume arclength parametrization by $(0,\ell)$. At smooth points of $\tilde\gamma$ we consider the unit conormals $\tilde\eta_j := J\dot{\tilde\gamma}_j \in T\widetilde{S_j^+}$. To represent the periods of the two different cousin curves $\gamma_j \subset S_j$ we observe that by the left invariance of the metric on $\mathbb{S}^3$ we have $\langle \tilde\eta_j, \mathbf{k}\rangle = \langle \dot\gamma_j, e_z\rangle$. Hence

$$P_j(i) = \gamma_j^3(l) - \gamma_j^3(0) = \int_{(0,l)} \langle \dot\gamma_j, e_z\rangle = \int_{(0,l)} \langle \tilde\eta_j, \mathbf{k}\rangle, \qquad j = 1, 2,$$

where $\gamma_j^3(l)$ is the z-component of $\lim_{t\to l}\gamma_j(t)$.

Except at isolated points, two nonidentical minimal surfaces intersect along a curve transversally, or with different conormals. From our assumption that $\tilde S_1 \cap \tilde S_2$ is empty to one side of $\tilde\gamma$ we conclude that $\langle \tilde\eta_1 - \tilde\eta_2, \mathbf{k}\rangle$ has the same sign on a dense subset of $(0,l)$. The claim follows. □

The lemma allows us to apply the maximum principle to our noncompact cousin disks.

Theorem 3.7. *The classifying map* $\Psi\colon \mathcal{M} \to \mathcal{T}$ *is injective.*

Proof. Consider $M, N \in \mathcal{M}$ with $\Psi(M) = \Psi(N) = T \in \mathcal{T}$; let n_1, n_2, n_3 denote the necksizes of M or N. We regard the cousins $\widetilde{M^+}$ and $\widetilde{N^+}$ as sections of $E(T)$.

The set $R_i \times \mathbb{R} \subset E(T)$ is foliated by spherical helicoids, namely all vertical translates of the cousin of a half unduloid with necksize n_i. By the asymptotics result of [KKS] and Lemma 1.5, each cousin is asymptotic to some helicoid leaf over R_i, and so a vertical translation makes $\widetilde{M^+}$ asymptotic to $\widetilde{N^+}$ over the first end R_1.

We now claim that $\widetilde{M^+}$ is also asymptotic to $\widetilde{N^+}$ over the other two ends R_2 and R_3. We say M *sits above* N *over* $\Omega \subset D$ if M has a larger projection to the line fiber of E than N for any point in Ω. If the claim does not hold we can vertically translate $\widetilde{N^+}$ so that on the one hand all ends of $\widetilde{M^+}$ and $\widetilde{N^+}$ are asymptotic to different helicoid leaves, and on the other hand $\widetilde{M^+}$ sits above $\widetilde{N^+}$ on one end, say the first one, while on the remaining two ends it sits below. Let us then consider the open set $D' \subset D$ containing the points of D such that $\widetilde{M^+}$ sits above $\widetilde{N^+}$. By assumption, D' has a component D'' which is a neighbourhood of e_1. Then $\partial D'' \subset D$ is a piecewise smooth curve from q_1 to q_2 of finite length. Moreover, $\widetilde{M^+}$ and $\widetilde{N^+}$ do not intersect over D''. So we can apply Lemma 3.6 to find that the periods of M^+ and N^+ cannot simultaneously vanish. This contradicts the fact that M^+ and N^+ are triunduloid halves.

As the sections are asymptotic at infinity, and have the same smooth $\mathbf{k}$-lines as boundary we can use the interior-or-boundary maximum principle to show that $\widetilde{M^+} = \widetilde{N^+}$. □ Could we put in a reference for this version of the maximum principle? –jms

4. Properness

Korevaar and Kusner [KK] prove that each embedded k-unduloid of genus g is enclosed in a union of solid cylinders and spheres, the radius and number of which only depends on g, k. Making use of this enclosure theorem they derive area and curvature estimates.

We want to get similar estimates for triunduloids, which may only be Alexandrov embedded. The enclosure theorem and area estimates from [KK], and their proofs, carry over literally to this setting: There is a constant $C > 0$ such that for any triunduloid M, we have $\operatorname{area}(M \cap B_r) \leq Cr^2$.

However, this is not the case for the curvature estimate. One step of its proof relies on the theory of embedded minimal surfaces of finite total curvature. Here, we use instead certain special properties of Alexandrov-symmetric minimal surfaces.

Which norm is it, anyway? –jms

We write $|A|$ for the pointwise norm of the second fundamental form of a surface M.

Lemma 4.1. *Suppose* $\sup_{M_i} |A| \to \infty$ *for a sequence* M_i *of triunduloids. Then there are points* $p_i \in M_i$ *such that the rescaled surfaces* $M_i' := |A(p_i)|(M_i - p_i)$ *have a subsequence converging to an Alexandrov-symmetric (nonplanar) genus-zero minimal surface* N *of finite total curvature.*

Here, convergence is in the sense of local $C^{2,\alpha}$-graphs. The limit can have positive multiplicity, but this must be finite by the area bounds.

Proof. We choose the points $p_i \in M_i$ such that on M_i' we have $|A| \leq 2$ on balls $M_i' \cap B_{R(i)}(0)$, whose radii satisfy $|A(p_i)| \geq R(i) \to \infty$ (see [KKS, p.487]). Due to this uniform curvature bound and the area bound, the M_i' subconverge to a minimal surface N. On N, we have $|A(0)| = 1$, so N is not a plane.

To bound the total curvature of N, we follow [KKS] in choosing connected subsets $S_i \subset M_i$ with uniformly bounded area such that $M_i \cap B_1(p_i) \subset S_i$ and ∂S_i consists of geodesic loops (around necks of the surface). Since the M_i are triunduloids, the number of loops needed is at most three; the Gauss-Bonnet theorem then uniformly bounds the total curvature of S_i. Because total curvature is invariant under rescaling, and the rescaled S_i converge to N, we get that N has finite total curvature $\int_N |A|^2$. (See [KK, §5] and [KKS, p.486–488] for details.)

The M_i are Alexandrov-symmetric with respect to a horizontal plane P. Suppose for a subsequence M_i the points p_i are not within distance $O(1/|A(p_i)|)$ of P. Then by the Alexandrov symmetry property, the Gauss map near p_i takes values in a hemisphere. By C^1-convergence, the minimal surface N has Gauss image contained in a hemisphere, but this contradicts the fact that N is nonplanar. Therefore, instead, the M_i' have their symmetry planes at a bounded distance from the origin; a subsequence will have convergent symmetry planes. By C^1-convergence, we can take N to have Alexandrov symmetry in this limit plane. □

To make use of this proposition, we now consider finite-total-curvature minimal surfaces. Each end has a well-defined normal and multiplicity [HK, Sect.2.1]. If an end is embedded, it has winding number one, and it is asymptotic to a catenoid or a plane, in which case we call it *catenoidal* or *planar*. (Ends of higher winding number may have Enneper type, or may multiply cover a catenoid or plane; these cannot be embedded.) We call an end *horizontal* when its limiting normal is vertical, and vice versa.

Lemma 4.2. *Let* N *be a minimal surface of finite total curvature, having Alexandrov symmetry with respect to a horizontal plane* P*. Then each end is either horizontal or* *vertical, and each vertical end is catenoidal. Furthermore, if $N \cap P$ contains an unbounded curve, then there is a simple closed geodesic on N, symmetric with respect to P, which has nonzero force.*

We need to recall the notion of forces (see [KKS, §3]). If Γ is a cycle on a surface M with constant mean curvature H, the vector $f(\Gamma) := \int_\Gamma \eta - 2H\int_K \nu \in \mathbb{R}^3$ is called the *force* of Γ. Here η is the outer conormal along Γ and K is a 2-chain with boundary Γ and normal ν. The first variation formula for area shows that the force depends only on the homology class $[\Gamma] \in H_1(M)$, so we write $f([\Gamma])$. Note that each annular end E of M defines a cycle, and thus has an associated force. (On a surface of finite topology, the sum of all these cycles around the ends is null-homologous; thus we get the *balancing formula*, which says that sum of the force vectors of all the ends is 0.)

I replaced weights by forces throughout for simplicity, and defined the balancing formula which we need in S.5. –jms

Note that an unduloidal or catenoidal end has force equal to that of the asymptotic unduloid or catenoid. This is directed along the axis, and by rotational symmetry can be calculated to have magnitude

$$n\left(1 - H\frac{n}{2\pi}\right),$$

where n is the necksize. In particular, the force a catenoidal end of a minimal surface has nonzero magnitude given by the necksize (or *logarithmic growth*) of the asymptotic catenoid.

Proof. An Alexandrov-symmetric surface meets the horizontal symmetry plane P perpendicularly. An end not meeting P must be horizontal; an end meeting P is vertical and itself Alexandrov-symmetric. At any planar end of a minimal surface, the Gauss map has a branch point [HK], so a vertical planar end contradicts Alexandrov symmetry. Similarly, an end with higher winding number cannot be decomposed into symmetric halves on the two sides of P, so it cannot be Alexandrov-symmetric.

check for precise reference

To prove the last statement, note that if N has no vertical ends then $N \cap P$ is a union of compact curves, contrary to our assumption. Thus, N has a vertical catenoidal end E, with nonzero force. Consider a simple closed loop around E, which is nontrivial in N. By the asymptotics of E, there is a unique simple closed geodesic γ homotopic to this loop. The reflection of γ in P is also a geodesic, and so coincides with γ. Thus γ is symmetric; its force equals that of E. □

Why is this unique? –jms

Proposition 4.3. *Given $\ell > 0$, there exists a constant $C(\ell)$ such that for any triunduloid M on which all nontrivial closed curves have length at least ℓ, the curvature is uniformly bounded:*

$$|A| \le C(\ell).$$

Proof. Suppose we have $|A| \to \infty$ for some sequence of triunduloids M_i, on each of which all nontrivial closed curves have length at least ℓ. Then by Lemma 4.1, they limit to an Alexandrov-symmetric minimal surface N. The M_i have genus 0, so there are no closed loops in $M_i \cap P$. Since the rescaled M_i converge to N with finite multiplicity, there are no closed loops in $N \cap P$ either. Since N does meet P, it does so in an unbounded curve. Thus, we can apply Lemma 4.2 to get a closed geodesic γ in N with nonzero force. This is the limit of closed curves γ_i' on M_i', whose lengths and forces converge to those of γ. So on the unrescaled M_i, the corresponding curves γ_i have nonzero forces, and are thus nontrivial. But their lengths converge to zero, contradicting our assumption. □

In the case of triunduloids the curvature estimate can be given instead in terms of a lower bound on the necksizes. We use this fact to prove the main result of this section.

Theorem 4.4. *The classifying map $\Psi\colon \mathcal{M} \to \mathcal{T}$ is proper.*

Why is this second theorem (3.2=thtrco) mentioned? –jms

Proof. Consider a compact subset $\mathcal{K} \subset \mathcal{T}$. By Theorems 2.1 and 3.5 there is an $\varepsilon > 0$ such that each surface $M \in \Psi^{-1}(\mathcal{K})$ has necksizes bounded below by ε.

We claim that any nontrivial closed curve on the triunduloid M has length at least ε. Indeed, M intersects its mirror plane P in three curvature lines. The complement in M of any two of these lines is simply connected, so a nontrivial closed curve in M must meet at least two of the lines. Thus its length is at least twice the distance in M^+ between these lines. But in the isometric surface $\widetilde{M^+}$ these are Clifford-parallel great circles; the distance between them is half a necksize of M, at least $\varepsilon/2$. Therefore, the curvature estimates of Proposition 4.3 hold uniformly on $\Psi^{-1}(\mathcal{K})$, proving the claim.

Let us now show that a sequence $M_i \in \Psi^{-1}(\mathcal{K})$ subconverges with respect to the topology on $\mathcal{M}$. The M_i are only given up to horizontal translation, but we want to fix a representative in $\mathbb{R}^3$ and get convergence there. One way to do this is to pick a point $p_i \in M_i^+$ contained in the embeddable triangle Δ of Lemma 3.4. We translate M_i such that p_i becomes the origin of $\mathbb{R}^3$. Due to the area and curvature estimates we find a subsequence of the M_i that converges on each compact subset to some CMC surface M. Since $0 \in M_i$ for all i, we have $0 \in M$ and thus M is nonempty.

Moreover, M has three ends. Indeed, reflect the curves on M_i corresponding to $\partial\Delta$ to form three symmetric closed curves. Since we chose p_i at the origin, the upper halves of these curves are contained in $B_\pi(0)$. The three families of curves limit to three curves in M; these carry nonzero forces because of the uniform necksize bound on $\mathcal{K}$. Finally, a sequence of contractible loops has a contractible limit. Thus M is topologically a sphere minus three points. It is clear that M is Alexandrov embedded. □

Add some detail here—we knew only certain things could have gone wrong in the limit, so it's enough to find 3 nontrivial curves. –jms

5. Surjectivity

5.1. **The real-analytic variety structure of $\mathcal{M}$.** The moduli space of CMC surfaces of any fixed finite topology is a real-analytic variety, as is shown by the local structure theory of Kusner, Mazzeo, and Pollack [KMP]. In order to discuss the regular points of this variety, we define a notion of nondegeneracy.

Definition. A (minimal or CMC) surface M is called *degenerate* if the Jacobi equation $\Delta_M u + |A|^2 u = 0$ has a nonzero square-integrable solution $u \in L^2(M)$. Otherwise M is called *nondegenerate*.

At a nondegenerate CMC surface of finite topology with k embedded ends, [KMP] use the implicit function theorem to show the asymptotic parameters given by the k force vectors of the ends locally determine the surface completely. The balancing formula and the action of $\mathsf{SO}(3)$ each reduce the number of parameters by 3 to $3k-6$. Thus for triunduloids, the nine parameters are reduced to three:

Theorem 5.1 ([KMP]). *The moduli space $\mathcal{M}$ is locally a real-analytic variety of finite dimension; in a neighborhood of any nondegenerate triunduloid, it is a manifold of dimension* 3.

We observe that our notion of a degenerate point does not agree with the one commonly used for varieties.

There are cases known where the variety dimension differs from $3k-6$. For instance the space of so-called multibubbletons with two ends has higher dimension. Similarly, the moduli space of CMC tori is known to have components of arbitrary positive dimension; these components are entirely degenerate.

On the other hand, it seems likely that most Alexandrov embedded surfaces, and perhaps all the Alexandrov-symmetric ones, are nondegenerate. While the methods of [G] give certain degenerate embedded triply periodic CMC surfaces (considered as embeddings of a compact surface into a flat 3-torus), no similar example is known with finite topology.

5.2. **Nondegenerate triunduloids.** The existence of the injective map $\Psi\colon \mathcal{M} \to \mathcal{T}$ shows that the variety $\mathcal{M}$ cannot have dimension greater than 3. However, to apply topological methods to get surjectivity, we need to know the dimension is exactly 3. To get this, we want to get the existence of a nondegenerate triunduloid, near which $\mathcal{M}$ will be a three-manifold.

Montiel and Ros [MR] have a theorem about nondegeneracy of minimal surfaces, which we specialize slightly here. A *minimal k-noid* is a minimal surface of finite total curvature with only catenoidal ends.

Theorem 5.2 ([MR, Cor. 15]). *Suppose M is a minimal k-noid, and the Gauss image of the umbilic points of M is contained in a great circle in $\mathbb{S}^2$. Then the only bounded Jacobi fields on M are those induced by translations in $\mathbb{R}^3$.*

In our formulation we have used the fact that the Gauss map of a k-noid does not branch at the ends.

Corollary 5.3. *All minimal trinoids are nondegenerate.*

Proof. Note that a trinoid M has total curvature -8π. Hence the Gauss map has degree two and by Riemann-Hurwitz at most two branch points, that is, umbilic points of M. Therefore the theorem applies. If M were degenerate, there would be an L^2 Jacobi field, which would be bounded, and thus induced by a translation. But a translation of a catenoidal end (even if orthogonal to the axis) fails to be L^2. $\square$

Mazzeo and Pacard [MP] find CMC surfaces by fusing the ends of a sufficiently small nondegenerate minimal k-noid with unduloid or nodoid ends of small necksize.

Theorem 5.4 ([MP]). *Let $M_0 \subset \mathbb{R}^3$ be a nondegenerate minimal k-noid that can be oriented so that each end has normal pointing inwards towards its asymptotic axis. Then for some ε_0 and $0 < \varepsilon < \varepsilon_0$, there is a family M_ε of nondegenerate CMC surfaces with k embedded ends, such that on any compact set in $\mathbb{R}^3$, the dilated surfaces $\frac{1}{\varepsilon}M_\varepsilon$ converge smoothly and uniformly to M_0.*

Gluing three unduloid ends to a minimal trinoid yields a surface which is Alexandrov embedded (though not embedded for small necksizes). Thus we can combine Corollary 5.3 with Theorem 5.4 to yield:

Corollary 5.5. *There exists a nondegenerate triunduloid.*

5.3. **Topological arguments for surjectivity.** We recall (see [Mu, §7-4]) that a space Y is *compactly generated* provided that any subset A is open (resp. closed) in Y if and only if $A \cap K$ is open (resp. closed) in K for every compact $K \subset Y$. Any locally compact space Y is compactly generated, as is any space Y which has a countable base at every point.

Lemma 5.6. *If $f\colon X \to Y$ is a continuous, proper, injective map from an arbitrary space X to a compactly generated Hausdorff space Y, then f is an embedding, that is, a homeomorphism onto its image $f(X)$. Furthermore, this image is closed in Y.*

Proof. Because f is a continuous injection, it is an embedding if and only if it is a closed (or open) map onto its image. Given a closed set $A \subset X$, we will show $f(A)$ is closed in Y, by showing $f(A) \cap K$ is closed for all compact $K \subset Y$. Since f is proper, $f^{-1}(K)$ is compact, and so the closed subset $A \cap f^{-1}(K)$ is also compact. Thus its image $f(A) \cap K$ is compact, and (since Y is Hausdorff) closed.

Applying this argument to $A = X$, we see that $f(X)$ is closed in Y. □

Definition. Let X be a locally finite d-dimensional simplicial complex and S a $(d-1)$-simplex in X. The *valence* of S is the number of d-simplices which contain S as a face. The complex X is *borderless* if no $(d-1)$-simplex has valence 1.

Theorem 5.7. *If $f\colon X \to Y$ is a continuous, proper, injective map from a borderless d-complex X to a connected d-manifold Y, then f is surjective and thus (by the lemma) a homeomorphism.*

Proof. Consider the union of the closed d-simplices in X and remove its $(d-2)$-skeleton. Let X' be the closure (in X) of some connected component of this set. Its $(d-1)$-simplices have valence at least two because X is borderless. Since X' is closed, the restriction $f|_{X'}$ is still proper. We will show this restriction is surjective. (Note that implies in fact $X = X'$ and every valence is exactly two.)

By Lemma 5.6, f is an embedding and $f(X')$ is closed in Y. In particular, the image of the $(d-2)$-skeleton X'_{d-2} is a properly embedded $(d-2)$-complex in the connected d-manifold Y; therefore its complement B is connected. Letting Z be the complement of X'_{d-2} in X', we have $f(Z) = B \cap f(X')$ is a closed subset of B. We claim that $f(Z)$ is also open in B. It then follows by the connectedness of B that f maps Z onto B. Again by the lemma, $f|_{X'}$ is closed, and so f maps $\overline{Z} = X'$ onto $\overline{B} = Y$, as desired.

Let us now prove the claim. Suppose first that $z \in Z$ is interior to a d-simplex. Since the dimension of Y is also d, and f is an embedding, $f(Z)$ contains an open neighborhood of $f(z)$. Otherwise z is interior to a $(d-1)$-simplex S. Because X' is borderless, z is in the interior of the union of two d-simplicies with common face S, so again $f(Z)$ contains a neighborhood of $f(z)$. (Note that the injectivity of f also shows the valence of S is exactly 2.) □

Definition. Let X be a d-dimensional locally finite simplicial complex. We say X is a *(mod 2) Euler space* if the link of each simplex in X has even Euler number.

The *link* of a $(d-1)$-simplex S is obtained as follows. Consider the vertices of the d-simplices with S as a face; those which are not in S itself form the link of S. The valence of S gives the number of points in the link. For a (mod 2) Euler space the link of any $(d-1)$-complex consists of an even number of vertices. Therefore, a (mod 2) Euler space is borderless.

Łojasiewicz [Ło] showed real-analytic varieties are triangulable as locally finite simplicial complexes, and Sullivan [S, BV, H] discovered that they have the (mod 2) Euler property. As a consequence of these results, we have:

Proposition 5.8. *A real analytic variety, or any space locally homeomorphic to one, is a borderless complex.*

Making use of our previous results on injectivity and properness we can now prove:

Theorem 5.9. *The classifying map $\Psi\colon \mathcal{M} \to \mathcal{T}$ is a homeomorphism.*

Proof. To apply Theorem 5.7, we need to verify its assumptions. Ψ is proper and injective by Theorems 4.4 and 3.7. From Theorem 5.1 the moduli space $\mathcal{M}$ of triunduloids is locally a real analytic variety, which is is borderless by Proposition 5.8. The three-manifold $\mathcal{T}$ is connected.

It remains to check that $\mathcal{M}$ also has dimension 3. Since the classifying map Ψ is injective from $\mathcal{M}$ to the 3-manifold $\mathcal{T}$, there are no simplices of dimension $d > 3$ in $\mathcal{M}$. By Corollary 5.5 there is a nondegenerate triunduloid, so by Theorem 5.1 $\mathcal{M}$ is a 3-manifold in some neighborhood of this triunduloid. □

Together with Theorem 2.1 this concludes the proof of our main theorem.

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