Conservation Laws and Nonlinear Hyperbolic PDEs

Rob Chase and Pat Dragon
Under Supervision of Robin Young
Motivation

• Traffic Patterns
  – Cars are neither created nor destroyed

• Astronomy
  – Spiral Galaxies
  – Supernovae

• Fluid Dynamics
  – Shallow water wave equations
  – Euler’s gas dynamic equations
Euler’s Shocktube

A long thin tube that allows movement of conserved quantities only in one dimension. (think of a highway)
## ODEs vs PDEs

**ODEs involve derivatives with respect to only one variable**

- \( x' = 4x \) (linear)
- \( y' = 4y^2 \) (nonlinear)

**PDEs involve partial derivatives with respect to space/time**

- \( u_t + u_x = 0 \) (linear)
- \( u_t + u^*u_x = 0 \) (nonlinear)
Systems of PDEs

\[ \begin{align*}
    u_t + f^1_x(u,v,w) + g^1_y(u,v,w) + h^1_z(u,v,w) &= 0 \\
    v_t + f^2_x(u,v,w) + g^2_y(u,v,w) + h^2_z(u,v,w) &= 0 \\
    w_t + f^3_x(u,v,w) + g^3_y(u,v,w) + h^3_z(u,v,w) &= 0
\end{align*} \]

Define \( U = \text{Transpose}(u,v,w) \)

\[ \begin{align*}
    U_t + F_x(U) + G_y(U) + H_z(U) &= 0 \\
    U(0,x,y,z) &= \text{Initial Conditions}
\end{align*} \]
Two representations of initial conditions:

Initial conditions as profiles at time $t=0$

Initial conditions as a curve in statespace parameterized by $x$
Hyperbolicity

A system is called *hyperbolic* if the flux matrix $F$ has real eigenvalues.

A hyperbolic system is called *strictly hyperbolic* if the real eigenvalues are all distinct.

If the eigenvalues are distinct, then the eigenvectors are independent.
Characteristic Curves

Analogous to level curves of surfaces in 3D.

In linear systems, the characteristics are parallel.

In some nonlinear systems, the characteristics intersect forming discontinuities and waves.

Characteristics are straight lines unless they interact with waves.
Finding the Eigensystems of PDEs

The eigensystem of a flux matrix may be calculated using linear algebra. Finding the eigensystem, the system may be “decoupled” into separate equations for each state variable. The resulting system of ODEs is easier to solve.
This Summer...

\[ v_t + f_x(v) = 0 \]

\[ W = \text{Transpose}(u, z) \]

\[ W_t + [A(v)W]x = 0 \]

The eigensystem of \( A \) can be used to find the 3x3 eigensystem.

Maintaining Strict Hyperbolicity we will find flux functions and initial data that will “blow up” but remain “smooth.”