There are two questions that have naturally come up in studying differential calculus and the definite integral:

- does \( f \) have an antiderivative (a function \( F \) such that \( F' = f \)), and
- is there an easier way to calculate the definite integral of \( f \) over an interval.

I know, for example, that the cosine function has an antiderivative, but what about the cosecant? And while one can struggle through calculating a definite integral like
\[
\int_0^1 x^2 \, dx
\]
using Riemann sums, I don’t want to compute even something like \( x^5 \), let alone something like \( e^x \).

These questions come up in very different contexts, but they have the same answer:

**Theorem** (Fundamental Theorem(s) of Calculus). Let \( f \) be a continuous real-valued function defined on a closed interval \([a, b]\).

(a). Then the function
\[
G(x) = \int_a^x f(t) \, dt,
\]
defined for all \( x \in [a, b] \), is continuous on \([a, b]\), differentiable on \((a, b)\), and \( G'(x) = f(x) \) for all \( x \in (a, b) \).

*Describe what \( G \) looks like* Equivalently, one sometimes writes this as
\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

(b). If \( F \) is an antiderivative of \( f \) on \([a, b]\), then
\[
\int_a^b f(x) \, dx = F(b) - F(a) = F(x) \bigg|_a^b.
\]

Equivalently, one sometimes writes this as
\[
\int_a^b \frac{dF}{dx} \, dx = \int_a^b F'(x) \, dx = F(b) - F(a)
\]

Let’s prove that the first implies the second.

**Proof.** Suppose \( F \) is an antiderivative of \( f \) on \([a, b]\). Since \( G \) is also an antiderivative of \( f \) on \([a, b]\), there exists some constant \( C \) such that \( G(x) = F(x) + C \) for all \( x \) in \([a, b]\) [Why?]. Plugging in both sides at \( a \), we see that
\[
F(a) + C = G(a) = \int_a^a f(t) \, dt = 0
\]
so \( C = -F(a) \), and plugging in at \( b \) we get
\[
G(b) = \int_a^b f(t) \, dt = F(b) + C = F(b) - F(a),
\]
which is what we wanted to show.

As an example of the second part, notice that I can compute very quickly
\[
\int_0^1 x^2 \, dx = \frac{x^3}{3} \bigg|_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3} \quad \text{and} \quad \int_0^{\pi/3} \cos(x) \, dx = \sin(x) \bigg|_0^{\pi/3} = \sin(\pi/3) - \sin(0) = \frac{\sqrt{3}}{2}.
\]
As an interesting example of what the first part tells you, let’s imagine that you never met the natural logarithm. We know that
\[ \frac{x^{n+1}}{n+1} \text{ is an antiderivative of } x^n \text{ for } n \neq -1 \]
but without the logarithm, there is no function whose derivative is \( x^{-1} \). So let’s define
\[ L(x) = \int_1^x \frac{1}{t} \, dt \text{ for all } x \text{ in } (0, \infty). \]
Then \( L'(x) = x^{-1} \) for all \( x \) in \((0, \infty)\). What else can we determine about \( L \) from this definition?

- Is \( L(x) \) increasing or decreasing? Why? What about its concavity?

- This means that \( L(x) \) is one-to-one, so it has an inverse \( E(x) \). Using the chain rule:
\[ x = L(E(x)) \quad \text{so} \quad 1 = L'(E(x))E'(x) = \frac{1}{E(x)}E'(x) \]
and thus \( E'(x) = E(x) \), or \( E \) is its own derivative.

- Let \( y \) be a positive real number, and consider the function \( L(xy) = \int_1^{xy} \frac{1}{t} \, dt \). Then
\[ L'(xy) = \frac{d}{dx} \int_1^{xy} \frac{1}{t} \, dt = \frac{1}{xy} y = \frac{1}{x} \]
so the functions \( L(xy) \) and \( L(x) \) have the same derivative. This means \( L(xy) = L(x) + C \) for some constant \( C \). Plugging in at 1,
\[ L(y) = L(1) + C = \int_1^1 \frac{1}{t} \, dt + C = 0 + C = C \]
so \( L(y) = C \), and we have just proved that \( L(xy) = L(x) + L(y) \).