Problem 1. Find the solution of the inhomogeneous ODE
\[ y'' - 6y' + 9y = 1 + 9t + e^{3t} \]
with initial conditions \( y(0) = y'(0) = 0 \).

Problem 2. Find the solution of the inhomogeneous ODE
\[ y'' + 3y' + 2y = e^{-t} + \cos(2t) \]
with initial conditions \( y(0) = y'(0) = 0 \).

Problem 3. Consider the inhomogeneous ODE
\[ y'' + 16y = r(t) \]
modelling a mass-spring system with \( m = 1, \gamma = 0, k = 16 \) and external driving force \( r(t) \).

(i) What is the undamped/intrinsic frequency of the system?
(ii) If we apply an exponentially decaying (over time) external force \( r(t) = e^{-2t} \) to the system, find the solution with initial condition \( y(0) = 1, y'(0) = 0 \).
(iii) If we apply a periodic external force \( r(t) = \cos(\omega t) \) answer the following:
   - Which value of \( \omega \) corresponds to resonance?
   - Calculate the solution with initial condition \( y(0) = y'(0) = 0 \) in the non-resonance case. What is the maximal amplitude of your solution when \( \omega = 3.8 \) (how does this amplitude compare in size to the amplitude of the external force?) and at which times is that maximal amplitude attained?
   - Calculate the solution in the resonance case for the initial condition \( y(0) = y'(0) = 0 \).
   - Are the oscillations in the resonance case as compared to the non-resonance case in phase or out of phase (and by how much approx.)?

Problem 4. Consider the inhomogeneous ODE
\[ y'' + 6y' + 10y = \cos(\omega t) \]
modelling a mass-spring system with \( m = 1, \gamma = 6, k = 10 \) and periodic external driving force \( \cos(\omega t) \) (which has amplitude 1).

(i) Find the solution of the ODE with initial condition \( y(0) = y'(0) = 0 \). Do you have to worry about the exceptional case in the table?
(ii) Express your solution in amplitude-phase form (you have to do this separately for the homogeneous \( y_H \) and particular \( y_P \) solutions.
(iii) How long does it take for the amplitude of \( y_H \) to decay (in absolute value) to below \( 10^{-5} \) (below which we assume \( y_H \) cannot be measured anymore)? Call this time \( t^* \). (continuous on next page)
(iv) After time $t^*$ which part of your solution $y(t)$ is still measurable? What is the maximal amplitude expressed in terms of the external frequency $\omega$?

(v) Which numerical value of $\omega$ makes this maximal amplitude largest? (This could be called the “damped resonance frequency” of the system).